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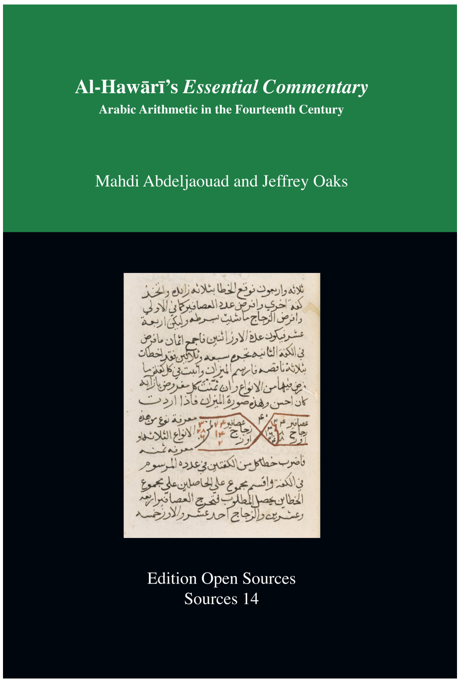
Translation

DOI: 10.34663/9783945561638-04



In: Mahdi Abdeljaouad and Jeffrey Oaks: *Al-Hawārī's Essential Commentary : Arabic Arithmetic in the Fourteenth Century*
Online version at `https://edition-open-sources.org/sources/14/`

ISBN 978-3-945561-63-8, DOI 10.34663/9783945561638-00
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The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at `http://dnb.d-nb.de`

Translation

Note: The Istanbul and Tunis manuscripts differentiate between passages attributed to Ibn al-Bannāʾ and those attributed to al-Hawārī, while the other three manuscripts we consulted do not make this distinction. In our translation, passages attributed to Ibn al-Bannāʾ that are taken from his *Condensed Book* are in **bold font**, while passages attributed to al-Hawārī are not in bold font. Passages borrowed from Ibn al-Bannāʾ's *Lifting the Veil*, often with minor changes in wording, are in italics: ***bold italics*** for those attributed to Ibn al-Bannāʾ, and *regular italics* for those attributed to al-Hawārī. Passages attributed to Ibn al-Bannāʾ that are neither in the *Condensed Book* nor in *Lifting the Veil* are in SMALL CAPS. Words and phrases added by us to b make the meaning clearer are placed in [square brackets]. Arabic words behind our translation are placed in parentheses, like "cube (*mukaʿʿab*)".

59.1 In the name of God, the Merciful and Compassionate, the humble and submissive servant to his Lord, hoping for His reward and His pardon for his sin, ʿAbd al-ʿAzīz ibn ʿAlī ibn Dawūd al-Hawārī al-Miṣrātī – may God forgive him – said: Praise be to God, the provider of graces and creator of life, who brings things into existence from nothingness, with praises the counting of which is uninterrupted. And full prayers on Muḥammad, His prophet and servant, and for His acceptance of the ancestors, who followed the revealed sunnas while not overstepping their boundary. And a prayer of supplication to our lord, the Commander of the Muslims, son of the Commander of the Muslims, Abū Yaʿqūb, deliverer of cherished victories from God. Then, may God prolong the term [in office] of our sovereign minister, the exalted, the radiant, the noble, the blessed, the honored, the jurisprudent, the learned, the all-knowing, the venerated of lofty virtue and superior intention, the superior, the proud, the issuer of laws guided by the light of the auspicious Marinid state, Abū Muḥammad ʿAbdallāh, son of our magistrate, the Shaykh Jurist, the virtuous, the pious, the ascetic, of excellent conduct and behavior, the late, sanctified, purest Ibn Madyan, who was made one of God's intercessors at the Gathering Place. May God perpetuate their memories among the pious, and protect the era, their return, and their place of honor from disgrace.

59.15 I begged almighty God to help me in my task of commenting on the book named *Condensed Operations in Arithmetic*, written by the incomparable scholar, our most eminent and open-minded master, Sheikh Abū l-ʿAbbās Aḥmad ibn Muḥammad ibn ʿUthmān al-Azdī, may God continue to bestow his divine blessing, care, and grace on him. I enhance my work with your name and embellish it with the splendor of your merits, so that I may deserve your blessing and your protection. To this end I asked our above-mentioned master, the scholar Abū l-ʿAbbās, for permission to undertake this work, to which request he kindly assented. Since the book I undertake to comment on has already been enriched by its own author, God bless him, with *Lifting the Veil*, containing all that is needed but providing only a few examples, I shall, almighty God willing, illustrate them with examples in the right place and when necessary. I call it the *Essential Commentary on the Condensed [Book] on the Operations of Arithmetic*. With God's help and assistance in all circumstances, it is time for me to begin the work, imploring the Almighty to guide and bless my endeavor.

61.1 **Our Master, the jurisprudent, the teacher, the leader, the learned, the radiant, the all-knowing, the guardian [of religion], Abū Aḥmad ibn Muḥammad ibn ʿUthmān al-Azdī, may God forgive him, said:**

61.3 **The goal of this book is to condense the operations of arithmetic, and to bring within reach its sections and its concepts, and to master its rules and its structures. It is comprised of two parts: the first is on operating with known numbers, and the second is on the basic rules through which the access to knowledge of the required unknown from the posited known is established, whenever a connection between them is provided. And I ask of God, praise be to Him, for assistance and success, and guidance toward the correct path.**

61.9 **Part One, on known numbers, which is divided into three chapters. The first is on working with whole numbers, the second on working with fractions, and the third on working with roots.**

63.1 **Chapter One, on whole numbers. For our purpose, this chapter is divided into six sections.**

65.1 **Section One, on the divisions of numbers and their ranks.**

65.2 **A number is a collection of units, and it is divided according to how it is produced into two kinds: whole and fractional.** Examples of whole numbers are fifteen, eighteen, and the like. Examples of fractions are a half, three eighths and half an eighth, a ninth and a fourth, six sevenths of seven eighths, and five sixths less a ninth.

65.6 **Whole numbers come in two varieties: even and odd.** An even number begins with a two or a four or a six or an eight, or does not begin with units, such as ten or fifty or the like. An odd number begins with a one or a three or a five or a seven or a nine.

65.10 **Even numbers come in three species: evenly-even, evenly-odd, and evenly-evenly-odd.** As for the evenly-even number, it is any number that can be halved, and each of its halves can be halved, until the halving reaches one. For example: thirty-two can be halved, and its half is sixteen. And each of the sixteens can be halved, and its half is eight. Half of the eight is four, half of the four is two, and half of the two is one, and the same for similar examples.

65.17 As for the evenly-odd number, it is any number that can first be halved and gives an odd number other than one. For example, fourteen can be halved, and its half is seven, which is an odd number other than one. The same goes for similar numbers. For this reason the [number] two is of the first species. So know it.

66.1 An evenly-evenly-odd number is any number that can be halved in such a way that each of its halves can be halved repeatedly until one reaches an odd number other than one. For example, twenty-eight can be halved, and its half is fourteen. Each of these fourteens can be halved, and its half is seven, which is an odd number, as we saw before. Since it can first be divided into evens, it may look like an evenly-even number, and since it reaches an odd number other than one, it may look like an evenly-odd number. So think it over.

66.7 **Odd numbers come in two species: prime and oddly-odd.** As for the odd prime number, it is any number that cannot be counted except by one, like eleven and twenty-nine and the like. They are also called deaf parts or simple, which will become clear in the work on the sieve.[1] As for the oddly-odd number, it is any number that can be counted by odd numbers, like fifteen, which is composed from the product of three by five. The same for similar examples.

66.13 I say we need to provide here an introduction in which we mention the names of the composite numbers according to their differences, and we will give examples of them. We say that the number, in relation to its composition, is either even or odd. Even numbers come in two varieties, either non-composite simple prime, which is uniquely two, or composite, which are of three kinds.

66.17 The composition of two equal numbers is called a square, or a number that has a root, and each of the two numbers is called a side or a root. For example, thirty-six is composed from six by six. The whole thirty-six is called a square or a number that has a root. Each of the sixes is called a side or a root.

67.3 The composition of two or more different numbers is called a surface, and each of these numbers is called a side. An example of the composition of two different numbers is eighteen, which is composed from three by six, or two by nine. The whole eighteen is called a surface. Each of the two and the nine, or the three and the six, is called a side. An example of the composition of three numbers is twenty-four, which is composed from three by four by two. So the whole twenty-four is called a surface, and each of the three and the four and the two is called a side. Likewise for more [numbers].

67.12 And the composition of three equal numbers is called a cube, and each of these numbers is a side and a cube root. For example, sixty-four is composed from the product of four by four by four. The whole sixty-four is called a cube, and each of these fours is called a side or a cube root. Likewise for similar situations. Some call the cube (*muka‘‘ab*) a cube root (*ka‘b*), which is a name for its side.

67.17 Odd numbers also come in two varieties: either simple, as we saw before, or composite, which are of three kinds, like the even numbers:

67.19 The composition of two equal numbers is called a square, or a number that has a root, and each of these numbers is a side or a root. For example, twenty-five is composed from five by five.

68.1 The composition of two or more different numbers is called a surface, and each of these numbers is a side. An example of the composition of two different numbers is thirty-five, composed from five by seven. An example of the composition of three different numbers is one hundred five, composed from three by five by seven. Likewise for more [numbers].

[1] The work on the sieve begins at 127.10.

68.8 The composition of three equal numbers is called a cube, and each of these numbers is called a side, or cube root. For example, twenty-seven is composed from three by three by three.

68.11 Subsection. The side and the cube root have come to have the same meaning, just as the root and the side also have the same meaning. They differ only as general and particular [substances] differ formally, but they are equal in usage. Taking a side of the cube[2] is a lengthy work of little benefit, which is why he[3] did not mention it, may God be satisfied with him. However, it can be found by means of decomposition, which is easy. You decompose the cube into the numbers of which it is composed. With them, you piece together three equal numbers by composition such that one of them is the required side. Study this with attention. You shall attain your aim, by God's will.

68.18 **As numbers increase indefinitely, they are placed in three ranks.** *They are called ranks because one of them follows another, and the units in each rank are greater than the units that came before them and smaller than the units that follow.*[4] **They are also called places,** because the number resides in them. The places of each number repeat periodically.

69.2 In each rank are nine numbers. The first rank consists of one to nine, and is called the units rank. Their figures are 1, 2, 3, 4, 5, 6, 7, 8, 9. Someone has written a poem about them:

> *Alif* and *ḥā* then *ḥajja* followed by *ʿuw*
> and after the *ʿuw* by an *ʿayn*. Draw
> *Hā* followed by a distinct figure
> looking like an anchor, and also you position
> Two zeros for eight with an *alif* [between them]
> and *wāw* is the ninth [digit]; so understand it.

69.9 The poet is right. He is also being clever when he states that eight is in the form of two zeros, since this helps us know the [figure for] zero.

69.10 **The second [rank] consists of ten to ninety, and is called the tens rank.** Their figures are 10, 20, 30, 40, 50, 60, 70, 80, 90. **And the third consists of one hundred to nine hundred, and is called the hundreds rank.** Their figures are 100, 200, 300, 400, 500, 600, 700, 800, 900.

69.15 **The names of numbers are formed from twelve simple names.**[5] **The first nine are those of the units, the tenth is for the tens, the eleventh is for the hundreds, and the twelfth is for the thousands, which is the units place**, in that it is the first of the three ranks, just as the units were first before its tens and its hundreds. And from there the cycle begins again.

70.2 And of the fourth rank, which is the thousands, we say: units, tens, hundreds, which all are thousands, and together they differ with the first three ranks only by the word "thousands".

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[2] I.e., calculating a cube root.
[3] Ibn al-Bannāʾ.
[4] Copied from (Ibn al-Bannāʾ 1994, 212.9-10).
[5] Literally, "Number has twelve simple names from which all of its names are formed".

Likewise for the three third ranks, which are thousands of thousands: they are also units and tens and hundreds, and differ with the preceding only by the word "thousand" twice. And likewise for the three fourth ranks, which [differ] with the preceding according to the previous description. And likewise for succeeding numbers. So know it.

70.8 For example,[6] five and twenty and two hundred and four and eighty thousands and a hundred thousands and seven and sixty thousands thousands and three hundred thousands thousands and nine thousands thousands thousands, and its figure is 9367184225. The four and eighty thousands and the hundred thousands are the units, tens, and hundreds like the first three ranks, differing only by the particular word "thousands". Likewise, the seven and sixty thousands thousands and three hundred thousands thousands are units, tens, and hundreds, and they vary from the preceding in being thousands of thousands. Likewise for the nine thousands thousands thousands. They are units, tens, and hundreds,[7] which differ from the preceding only by the repetition of "thousands" three times. So know it.

70.15 **It should be known that each number can be indicated by its index and its name. The index is a term for the rank of the number. The index of the units is one,** since they are in the first rank. **The index of the tens is two,** since they are in the second rank. **The index of the hundreds is three,** since they are in the third rank, **and so on.**

70.23 Take, for example, five and twenty and seven hundred and four and eighty thousands. Its figure is 84725. You find the five, which are units as mentioned before, in the first rank, and that is its index and the index of the other units. Likewise, the twenty belongs to the tens, which is in the second rank, and that is its index and the index of the other tens. So if it were, for example, thirty or sixty or eighty, it could have replaced the twenty, since tens have index two and are in the second rank, as mentioned. Likewise, the seven hundred is in the third rank, and that is its index, and the index of other [hundreds]. Likewise the four thousands belongs to the fourth rank, which is its index, and the index of other [thousands]. Likewise the eighty thousands is in the fifth rank, which is its index, and the index of other [tens of thousands]. So understand this. The same holds for greater numbers and for comparable numbers.

71.6 **The name is the term for the number that occupies some rank. The name of the one is units, of two is tens, and of three is hundreds.** Take for example three and forty and a hundred. Its figure is 143. You find that it has three places. The index of the first is one, and the name of this one is units, since it is the rank of the units rank. The index of the second is two, and the name of this two is tens, since it is the rank of the tens rank. The index of the third is three, and the name of this three is hundreds, since it is the rank of the hundreds rank. So understand it.

71.12 **Subsection on knowing the index of the repeated number.**[8]

[6] From here to the end of the section we write the numbers in the order they are written in Arabic, and with the stated plurals of "thousand". Unlike thousands, hundreds are not made plural when there is more than one because the numbers "two hundred", "three hundred", up to "nine hundred" are single compound words in Arabic. After that, we switch back to the English order, and without the plural.

[7] They are only units, since there is only a 9.

[8] I.e., a number expressed with a repetition of the word "thousands".

71.13 **You multiply the number of repetitions by three, and you add to the result the index of the species of that number to get the required number.** And the repetition is the number of times you say "thousand".

71.16 For example, suppose someone said to us, "What is the index of ten thousands thousands?" You find the number of repetitions to be two. You multiply it by three, giving six. Likewise if the repetitions are more or less than two: you must always multiply it by three. Then you add to this six the index of the species of ten thousands thousands, which is two, since it was stated before that the units of the thousands thousands are in the units place and the tens are in the tens place, and the hundreds are in the hundreds place. So the sum is eight, which is the index of the given number. So know it. Its representation is made of seven zeros and a one, as in this figure: 10000000.

72.1 **Conversely, if you have a great many places and you want its name, then divide it by three. The division leaves you with three or less. The quotient is the number of repetitions of the number obtained in the remainder.**

72.4 For example, suppose someone said to you, "What is the name of a number falling in the tenth place?" You divide the ten, which is the number of places, by three. Our quotient is three, and the remainder is one. Likewise if the number of places is greater or smaller than ten: you must always divide it by three. The resulting three is the number of repetitions for the name of the remainder one. And the name of the remainder one is units. The tens rankis the index of units of thousands thousands of thousands, and its figure is 1000000000.

72.10 If the remainder from the division were two, it would be tens of thousands of thousands of a thousand, and if it were three, it would be hundreds of thousands of thousands of a thousand.[9] So know it.

73.1 **Section two, on addition.**

73.2 **Addition is the joining of numbers one to the other in order to express them with one expression. This is divided into five types. One of them is addition with no known relation, and the second is addition with a known disparity**[10], which are divided into two kinds.

73.7 *[The first is] a disparity in quality, which occurs when the numbers are in geometric progression. The disparity of the numbers consists of different numbers which are equal in quality in the sense that the ratio of one number to the next is equal to the ratio of a half or a third or something else.*[11] This kind comes in two types, since the ratio can either be the ratio of a half, which is what he means when he says "addition with a disparity like that of the squares of the chessboard and similar [problems]",[12] where the one in the first square is half of the two in the second square, and the two is half of the four in the third square, and so on to the end; or the ratio can be a third or a fourth or a seventh or any other ratio, which is what he means when he says "if the numbers have another disparity".[13] Examples of these will be presented later, almighty God willing.

[9] Literally, "tens thousands thousands thousand" and "hundreds thousands thousands thousand".
[10] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.2-5).
[11] Copied from (Ibn al-Bannāʾ 1994, 214.3-5).
[12] From 76.7 below.
[13] From 78.1 below.

73.17 *The second kind is a disparity in quantity, in which the numbers are in arithmetic progression, such as consecutive numbers with a disparity of one, or consecutive odd numbers with a disparity of two, and the like. So the numbers differ by equal numbers, and it is a difference in quality when considered as the ratio of one number to the next.*[14] This is what he means when he says "If the disparity of the numbers is a known number other than doubling".[15] He, may God be satisfied with him, said "known disparity" and not "known relation" because *whenever the word "relation" is uttered, it suggests the well-known geometric [progression].*[16]

74.1 **The third [type] is the addition of consecutive numbers, their squares, and their cubes, the fourth [type] is the addition of consecutive odd numbers, their squares, and their cubes, and the fifth [type] is the addition of consecutive even numbers, their squares, and their cubes.**[17]

74.4 Note: *Adding consecutive numbers, consecutive odd numbers, and consecutive even numbers are truly [conducted in] arithmetic progression. The specific aim of these last three types is to find sums of [consecutive] squares and cubes. He included in these [types] the sums of [consecutive] numbers, even though they were of the preceding type, because they are a foundation for adding the squares and the cubes, and because working with a particular case makes the approach to the general one easier.*[18]

74.9 **For addition with no known relation, the aim is for you to add a number made up of several digits to a similar number. You should put one of the addends on a line, and below it you put the other addend, with each place below its counterpart. Then you add each digit of one of the addends to its counterpart in the other. If there is no counterpart, then the answer is the addend, as if it had a counterpart. So the sum is the answer.**[19] **One can start adding from the first digits or the last, but choosing the first is most orderly.**[20]

74.17 Here is an example of adding from the first digit. Suppose we want to add four thousand forty-three to two thousand six hundred eighty-five. We write the addend on a line and the augend below it on another line parallel to it, as mentioned, as in this figure: $\begin{matrix}4043\\2685\end{matrix}$.

74.20 We add the five, which is in the first [place] of the lower line, to its counterpart in the upper line, which is the three, giving eight. We put it above them, since they are units of their species. Then we add the eight that is in the tens rank of the lower line to its counterpart in the upper line, which is the four, giving twelve. We put the two above them, since they are also the same species, and we add the ten, in the form of a one, to the six in the hundreds rank of the lower line, giving seven, since for every rank, the units which are after it are the tens before it.[21] Nothing corresponds to the seven in the upper line, so it is considered to be the sum of that rank and that of its counterpart as if it had something. We put the

[14] Copied from (Ibn al-Bannāʾ 1994, 214.13-15).
[15] From 79.1 below.
[16] Copied from (Ibn al-Bannāʾ 1994, 214.18-215.1).
[17] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.5-8).
[18] Copied from (Ibn al-Bannāʾ 1994, 226.5-7).
[19] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.9-14).
[20] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.15-16).
[21] I.e., the units in a particular rank are the tens for the rank before it.

seven above the zero. Then we add the two in the thousands rank of the lower line to its counterpart in the upper line, which is the four, giving six. We put it above them, and this completes the work. The sum is six thousand seven hundred twenty-eight, and this is the figure for that: 6728.

75.9 Here is another example, adding from the last rank. Suppose we want to add nine hundred seventy-eight to four hundred fifty-six. We put them down on two lines, as mentioned, as in this figure: $\begin{matrix}978\\456\end{matrix}$.

75.11 We add the four in the hundreds rank of the lower line to its counterpart in the upper line, which is the nine, giving thirteen. We put the three above them, and the ten, in the form of a one, after the three. Then we add the five in the tens rank of the lower line to its counterpart in the upper line, and that is seven, giving twelve. We put the two above them and we add the ten, in the form of a one, to the three that is above the addends, giving four. We replace it with it. Then we add the six in the units rank of the lower line to its counterpart in the upper line, which is the eight, giving fourteen. We put the four above them, and we also add the ten, in the form of a one, to the two that is above the addends, giving three. We replace it with it, and this completes the work. The sum is one thousand four hundred thirty-four, and the figure for that is 1434.

75.20 **The most one can gain by addition is one place.**[22] For example, suppose we want to add nine to nine. These are the maximum numbers that can occur in those two places on the two lines. So we say nine [added] to nine gives eighteen, and its figure is 18. The sum gains one place.

76.4 **To check the addition, you subtract one of the two lines from the answer. This leaves the other line.** For example, if we want to check this problem, we subtract the nine, which is one of the addends, from eighteen, which is the answer. This leaves the other nine. So understand.

76.7 **For addition with a disparity like that of the chessboard squares and similar [problems], a one [is placed] in the first square, then one proceeds by doubling from the first [square] to another assigned [square]. You add one to the one that is in the first square to get what is in the second square. Then you multiply that by itself, so the outcome is what is in the second square and what is before it, with an added one. Then you also multiply that by itself, so the outcome is what is in the fourth square and what is before it, with an added one.**[23] **Continuing, you also multiply the result by itself, and you double the squares,**[24] **until you reach the assigned [square], and you drop the one from the sum. The remainder is the required number.**

76.15 For example, suppose we want to add what is in sixteen squares arranged as described. We add one to the one in the first square, giving two. We multiply it by itself, giving four. This is what is in the second square and what is before it, with an added one. It is also what is in the third square alone. Because the places of the squares are like the places of the number, no doubt that if we multiply a place by another place, the index of the result is always the

[22] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 131.17).
[23] Ibn al-Bannāʾ copied 76.7-12 from Ibn al-Yāsamīn, (Zemouli 1993, 132.1-6).
[24] I.e., double the number of squares, like from the fourth square to the eighth square.

index of the two multiplicands[25] less one, according to what we will show in [the section on] multiplication.[26] So when we multiply what is in the second square by itself, it most certainly results in what is in the third. So know it.

76.22 Then we likewise multiply the four by itself, giving sixteen, which is what is in the fourth square and what is before it, with an added one. It is also what is in the fifth square alone. Then we multiply the sixteen by itself, giving two hundred fifty-six, whose figure is 256. This is what is in the eighth square and what is before it, with an added one, and it is also what is in the ninth square alone. Then we likewise multiply the two hundred fifty-six by itself, giving sixty-five thousand five hundred thirty-six, whose figure is 65536. This is what is in the sixteenth square and all of what is before it, with an added one, and it is also all of what is in the seventeenth square alone. So it is clear that each number in a square exceeds all of what is before it by one. So you drop the added one, leaving sixty-five thousand five hundred thirty-five, which is the required number. Do it the same way if the required [number of squares] is greater or smaller. So know it. This is the figure for the table:

128	64	32	16	8	4	2	1
32768	16384	8192	4096	2048	1024	512	256

77.9 **If the situation is different, then multiply the remainder by the first square to get the required number.**[27] A DIFFERENT SITUATION IS WHEN THE FIRST SQUARE IS SOMETHING OTHER THAN ONE.

77.11 For example, suppose we want to add what is in eight squares, and a four is in the first square. We suppose that a one is in the first square, and we find the sum as before, and we drop the one. This is what he meant when he said "the remainder". It yields two hundred fifty-five, and its figure is 255. We multiply it by the four that is in the first square, which gives one thousand twenty. Its figure is 1020, and it is the required number.

78.1 **If the numbers have another disparity, then multiply the smallest by how much the greatest exceeds it, and divide [the result] by the difference between the smallest and the number that follows it. Then add the result to the greatest. This gives the required number.**[28]

78.4 For example, suppose we want to add five numbers with a ratio of, say, two thirds, like sixteen, twenty-four, thirty-six, fifty-four, and eighty-one. We put them on a line and we put dots between them, as in this figure: 81 ∴ 54 ∴ 36 ∴ 24 ∴ 16. We multiply the smallest, which is the sixteen, by how much the eighty-one exceeds it, since it is the greatest of the numbers, and that is sixty-five. The result is one thousand forty, whose figure is 1040. Then we divide it by the difference between the sixteen and the number after it, which is the twenty-four, and that is eight. The result of the division is one hundred thirty, whose figure is 130. We add it to the greatest to get two hundred eleven, whose figure is 211, and it is the required number.

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[25] I.e., the sum of their indexes.
[26] At 104.1 below.
[27] This sentence is copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.6).
[28] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.7-9).

78.13 *This procedure is common to all numbers that are in a geometric progression, by generalization with the ratio of a half or anything else. The preceding [solution] is particular. And the particular is a basis for the technique as much as the general is a basis, since working it out in general instead of the particular becomes unbeneficial due to its prolixity. And from this general procedure it is clear to you that any number in the squares of the chessboard exceeds the sum of what is before it by one.*[29] So know it.

79.1 **If the disparity of the numbers is through a known number other than doubling, then multiply the disparity by the number of numbers less one. Adding the first number to the result gives the last of the numbers. Add it to the first, and multiply it by half of the number of numbers. It yields the required answer.**[30]

79.4 For example, suppose we want to add six numbers, the first being ten, which is the smaller extreme, with a disparity of three down to the last. There are two unknowns in this problem: the greater extreme, which is the last, and the sum. We multiply the three, which is the disparity, by the five, the number of numbers less one, giving fifteen. We add to it the ten, which is the first number, giving twenty-five, which is the last of the numbers. We also add it to the first, yielding thirty-five. We then multiply it by three, which is half of the number of numbers, giving one hundred five. Its figure is 105, and it is the total of these numbers, I mean their sum. In this way the sum is clear to you, and it is necessarily the way to do it. So know it.

79.13 **For the addition of consecutive numbers, you multiply half of the upper extreme by the upper extreme and one.**[31] For example, suppose we want to add from one to ten consecutively. We add one to the ten, the upper extreme, and that is eleven. We multiply it by half of the ten, giving fifty-five, which is the sum, and its figure is 55.

79.18 **And squaring them is given by multiplying two-thirds of the upper extreme increased by a third of one, by the sum.** For example, suppose we want to add from a square of one to a square of ten consecutively. We take two thirds of the ten, the upper extreme, giving six and two-thirds. We always add a third of one to it, giving seven. We multiply it by the sum, and that is fifty-five. The result is three hundred eighty-five, which is the required number, and its figure is 385.

80.1 **And cubing them is given by squaring the sum.** For example, suppose we want to add from a cube of one to a cube of ten consecutively. We multiply the sum, which is fifty-five, by itself. The result is three thousand twenty-five, which is the required number, and its figure is 3025.

80.5 **For the addition of consecutive odd numbers you square half of the established upper extreme with the one.** For example, suppose we want to add the odd numbers from one to nine consecutively. We add one to the nine, the upper extreme, giving ten. We then multiply half of the ten by itself, giving twenty-five, which is the required number, and its figure is 25.

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[29] Copied from (Ibn al-Bannāʾ 1994, 220.8-12).

[30] This passage belongs in the *Condensed Book*, but it is not in Souissi's edition. It is copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.9-12).

[31] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 136.13-14).

80.10 **And squaring them is given by multiplying a sixth of the upper extreme by the surface of the two numbers that come after it.** For example, suppose we want to add the odd numbers from a square of one to a square of nine consecutively. We multiply a sixth of the nine, the upper extreme, which is one and a half, by the surface of the ten by the eleven, and that is one hundred ten, since they are the two numbers which are after the nine. The result is one hundred sixty-five, which is the required number, and its figure is 165.

80.15 **And cubing them is given by multiplying the sum by its double less one.** For example, suppose we want to add the odd numbers from a cube of one to a cube of nine consecutively. We multiply the sum, which is twenty-five, by its double less one, which is forty-nine. The result is one thousand two hundred twenty-five, which is the required number, and its figure is 1225.

80.20 **For the addition of consecutive even numbers, you always add two to the upper extreme, and you multiply half of the sum by half of the upper extreme.** For example, suppose we want to add the even numbers from two to ten consecutively. We add two to the ten, the upper extreme, giving twelve. We multiply its half by half of the ten, giving thirty, which is the sum, and its figure is 30.

81.4 **And squaring them is given by multiplying two-thirds of the upper extreme and two thirds of one by the sum.** For example, suppose we want to add the even numbers from a square of two to a square of ten consecutively. We take two-thirds of the ten, the upper extreme, giving six and two-thirds, and we always add to it two thirds of one, giving seven and a third. We multiply it by the sum, which is thirty, so the result is two hundred twenty, which is the required number, and its figure is 220.

81.9 **And if you wish, multiply a sixth of the upper extreme by the surface of the two numbers that come after it.** For example, suppose we want to add from a square of two to a square of twelve. We take a sixth of the twelve, the upper extreme, giving two. We multiply it by the surface of thirteen by fourteen, which is one hundred eighty-two, since they are the next two numbers after the twelve. The result is three hundred sixty-four, which is the required number, and its figure is 364.

81.15 **And cubing them is given by multiplying the sum by its double.** For example, suppose we want to add the even numbers from a cube of two to a cube of ten consecutively. We multiply the sum, which is thirty, by its double, which is sixty. The result is one thousand eight hundred, which is the required number, and its figure is 1800.

82.1 For the preceding three types, ***whenever it begins with something other than one, you add from one to the upper extreme, then from one to the number before the beginning number. Then you drop the smaller from the greater. For the even numbers the two takes the place of the one.***[32] By understanding all of this and managing it you shall succeed, almighty God willing.

83.1 **Section Three, on subtraction.**

83.2 **Subtraction is the search for the remainder after the dropping of one of two numbers from the other. It comes in two types. [One] type is subtracting the smaller from**

[32] Copied from (Ibn al-Bannāʾ 1994, 228.13-15).

the greater one time. And [another] type is subtracting the smaller from the greater more than one time, until the greater vanishes or it leaves a remainder less than the smaller. This type is called testing by casting out.

83.6 **For the first type you should write the minuend on a line, and below it the subtrahend, arranged as in addition. You subtract each digit from its corresponding digit if you find it has a counterpart. If you do not find a counterpart, or if it is smaller than the subtrahend, then subtract the minuend from the subtrahend, and subtract the remainder from the next digit, and you then put the remainder in the place which agrees with its given rank.**

83.11 **If you wish, you can always add ten to the counterpart and subtract [the digit in the subtrahend] from the sum, and add one to the next digit of the subtrahend. Then continue in the same way until you have finished all of the subtrahend and minuend. You can start a subtraction from the first of the ranks or from the last. It is preferred to start from the last, contrary to what is preferred in addition.**

83.16 Here is an example of subtraction beginning from the last rank. Suppose we want to subtract four thousand nine hundred sixty-eight from five thousand thirty-five. We put them down on two parallel lines, like we mentioned for addition, as in this figure: $\begin{array}{c}5035\\4968\end{array}$.

83.19 We subtract the four which is in the thousands rank from its counterpart in the minuend, which is five. The remainder is one, so we put it above it. Then we likewise subtract the nine that is in the hundreds rank from its counterpart. There is nothing but a zero there, and the zero is nothing. So we subtract this nothing of the minuend from the nine of the subtrahend, leaving nine. We subtract it from the one that is above the five, since it is ten with respect to the zero, as mentioned. This leaves one. We put it above the zero, and nothing is above the five. Then we likewise subtract the six that is in the tens rank from its counterpart, which is three. [This is] smaller than the six of the subtrahend, so we subtract the three of the minuend from the six of the subtrahend, leaving three. We subtract it from the ten that is above the zero, leaving seven. We put it above the three, and nothing above the zero. Then we likewise subtract the eight that is in the units rank from its counterpart, which is five. [This is] smaller than the eight, so we subtract the five of the minuend from the eight of the subtrahend, leaving three. We subtract it from the seventy that is above the three. The remainder is sixty-seven. We put the seven above the five, and the sixty, in the form of a six, above the three. It is the remainder, and the figure for that is: $\begin{array}{c}0067\\ \hline 5035\\4968\end{array}$.

84.13 Here is another example of subtraction from the first [rank]. Suppose we want to subtract three thousand four hundred sixty-nine from six thousand five hundred forty-three. We put them down in two parallel lines, as mentioned, as in this figure: $\begin{array}{c}6543\\3469\end{array}$. We subtract the nine that is in the units rank from its counterpart in the minuend, which is three. [This is] smaller than the nine, so we add ten to the three of the minuend, giving thirteen. We subtract the nine from it. The remainder is four. We put it above the three, then we add one to the six that is in the tens rank of the subtrahend, giving seven. We likewise subtract it from its counterpart, which is four. [This is] smaller than the seven, so we add ten to the four, giving fourteen. We subtract the seven from it, leaving seven. We put it above the four. Then we add one to the four that is in the hundreds rank, giving five. We likewise subtract it from its counterpart, which is five, and it vanishes. So we put a zero above the five. Then we likewise subtract the three that is in the thousands rank – with nothing added to it, since nothing remains from the previous position – from its counterpart, which is the six. Three remains, and we put it above the six. This completes the work. The remainder is three thousand seventy-four, and the figure for that is 3074.[33]

85.8 **The most one can reduce by subtraction is one place.** For example, suppose someone said to you, “Subtract one from ten”. The remainder is nine. The minuend is reduced by a place. So understand it.

85.11 **To check the subtraction, you add the remainder to the subtrahend. The result is the minuend. Or you subtract the remainder from the minuend, leaving the subtrahend.** For example, in the above-mentioned problem the remainder is nine. We add it to the one that is the subtrahend, giving ten. This is the minuend. And if we subtract the remainder, which is the nine, from the minuend, which is the ten, there remains one, which is equal to the subtrahend.

85.16 In the first way ***of checking***[34] ***a subtraction by addition you should look for a number which, if you add it to the subtrahend, then [the sum] is equal to the minuend. One begins this from the first of the ranks***, just like those who work with *rūmī* signs do.

86.1 ***Whenever you are faced with subtracting a number from a number, and then the remainder from another number, and so on, then you work it out in the steps as indicated in this section.***

86.3 ***If we wish, we can piece the problem together with addition and subtraction. We collect the even terms from among the subtrahends, namely the second, the fourth, the sixth, and so on, and we add them together with the minuend. We likewise add the odd terms from among the subtrahends, namely the first, the third, the fifth, and so on, and we drop their sum from the first sum. The numbering with the subtrahends truly begins after the minuend [and continues] to the last.***[35]

86.9 *For example, subtract two from five, and the remainder from seven, and the remainder from eight, and the remainder from ten. Sometimes this is expressed with exclusions, so one says: ten less eight less seven less five less two.* We put it down in a line like this: 2 ∖ 5 ∖ 7 ∖ 8 ∖ 10. *We subtract the two from the five, and the remainder from the seven, and the remainder from the eight, and the remainder from the ten, leaving six, which is the required number.*

86.15 *If we wish, we can add the even subtrahends, which are the seven and the two, with the minuend ten, to get nineteen. Then we add the odd subtrahends, which are the eight and the five, giving thirteen. We subtract it from the nineteen, leaving six.*

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[33] The Medina and Oxford manuscripts show $\begin{array}{c}3074\\ \hline 6543\\ 3469\end{array}$.

[34] Literally, “working out”.

[35] Copied from (Ibn al-Bannāʾ 1994, 245.8-15).

86.18 *And if we wish, we can consider three successive terms. We subtract the middle from the sum of the extremes, leaving the remainder as one number.* We insert it in [their] place in the subtrahend. *Then we likewise consider it and the two remaining numbers. Take for example the ten and the eight and the seven. We drop the eight from seventeen, the sum of the extremes, leaving nine which, with the two and the five, are three successive numbers. We drop the five from the sum of the extremes, leaving six.*

87.3 *Or we first consider the seven and the five and the two. We drop the five from the nine, the sum of the extremes, leaving four which, with the eight and the ten, are three successive numbers. We drop the middle number, eight, from the sum of the extremes, leaving six.*

87.6 *Or we consider the eight and the seven and the five first. We drop the middle seven from the sum of the extremes, leaving six. This is now the middle term between the ten and the two. We drop it from their sum, leaving six.*

87.9 *And if we wish, we can subtract the eight from the ten, and add the remainder to the seven, and subtract the five from it, and add the remainder to the two, yielding six.*

87.11 *The cause of this is that when subtracting the deleted from the appended, it is deleted, but when subtracting the deleted from another deleted, it is appended. Thus the deleted second and fourth and sixth of the even terms are always appended, since each of them is deleted from a deleted. And the odd terms are always deleted, since they are deleted from an appended.*[36] So know it.

87.15 **The second type [of subtraction] consists of three [kinds of] subtractions. These are frequently used in checking work. One is casting out nines, the second is casting out eights, and the third is casting out sevens.**

87.17 **Casting out nines leaves one from each power of ten. You collect the number from its digits as if they were units, and then you cast out nines from them.**

87.19 For example, suppose we want to cast out [nines from] six thousand four hundred thirty-five. We put the number on a line like this: 6435. We add the five to the three, giving eight. We add it to the four, giving twelve. We subtract nine from it, leaving three. We add it with the six, giving nine. This is cast out entirely, and it is the answer.

88.1 **Casting out eights leaves two from each ten, four from each hundred, and a pair of hundreds and what is above it are cast out entirely. So four is the remainder from odd hundreds, and you multiply the tens by two,** SINCE THE REMAINDER FROM EACH TEN IS TWO**. You add that with the four and with the units, and you cast out eights from it.**

88.5 For example, suppose we want to cast out [eights from] five thousand three hundred ninety-three. We put down the number on a line like this: 5393. Five thousand is cast out entirely, and the remainder from the three hundred is four. Keep it in mind. Then we multiply the ninety[37] by two, giving eighteen. We add it to the remembered four with the three units, to get twenty-five. Casting out eights leaves one, which is the answer.

[36] Copied from (Ibn al-Bannāʾ 1994, 245.16-247.2).
[37] I.e., the nine, called "ninety" here because it is positioned one place forward.

88.10 **In casting out sevens, three remains from each ten, two from each hundred, six from each thousand, four from each ten thousand, five from each hundred thousand, one from each million, and from there the cycle begins again. You can come to know it with these letters: A J B W D H, repeated below the digits.**

88.14 The A is one, the J is three, the B is two, the W is six, the D is four, and the H is five. Someone arranged these elements into verse. He said, "Three and two and six and four * and five and one; that is casting out sevens".

88.17 IF YOU WISH, YOU CAN WRITE DOWN THE LETTERS AS MENTIONED, OR IF YOU WISH YOU CAN MAKE THEM DUST NUMERALS. **You multiply each digit by what is below it in number – sound out the letters – and you cast out sevens. You leave its residue above it. Then you add all the digits of the remainders as if they were units and you cast out sevens.**

89.4 For example, suppose we want to cast out [sevens from] twenty-three million seven hundred eighty-six thousand four hundred thirty-five. We write it down on a line and we draw a line above it. We write the letters below it, each letter below a number consecutively. If we run out of letters and we have not run out of digits, we repeat the letters for the remaining [digits]. This is what he meant when he said "and from there the cycle begins again". That is, after the H, and for the rest of the number, the letters are repeated along with the quantity with it. The figure always looks like this:

2	3	7	8	6	4	3	5
J	A	H	D	W	B	J	A

89.10 We multiply the five by the number of the A below it, giving five. We put it over the line above the five. Then we likewise multiply the three by the number of the J below it, giving nine. We cast out sevens, leaving two, and we put it over the line above the three. Then we multiply the four by the number of the B, giving eight, leaving one. We put it above the four. Then we multiply the six by the number of the W, giving thirty-six, leaving one. We put it above the six. Then we multiply the eight by the number of the D, giving thirty-two, leaving four. We put it above the eight. Then we multiply the seven by the H, giving thirty-five, which vanishes. So we put a zero above the seven. Then we likewise multiply the three by the repeated A, giving three. We do not cast out, so we put it above the three. Then we multiply the two by the J, giving six. We do not cast out here, either. We put it above the two. Its figure is:

6	3	0	4	1	1	2	5
2	3	7	8	6	4	3	5
J	A	H	D	W	B	J	A

90.1 Now we work with the remainders five, two, one, one, four, three, and six. We add them as if they were units, and we cast out sevens. The remainder is the answer, and that is one.

90.3 **If you wish, multiply what is in the last place by three. You cast out sevens, and you add the remainder to what is before it. If there is no number in the place before it, then you multiply the accumulated residue by three and you cast out sevens. Keep doing this until you reach the units [place].**

90.7 For example, suppose we want to cast out [sevens from] fifty-eight thousand sixty-four. We put it down on a line like this: 58064. Then we multiply what is in the last place, which is five, by three, giving fifteen. We cast out sevens, leaving one. We add it to the eight that is before it, giving nine. We multiply it by three, giving twenty-seven, whose remainder is six. There is nothing in the preceding place, so this six is considered to be the sum of the residue and the rank as if it had something. We multiply it by three, giving eighteen, and its remainder is four. We likewise add it to the six that is before it, giving ten. We multiply it by three, giving thirty, and its remainder is two. We add it to the four in the units place, giving six. This completes the work, and the six is the answer. So know it.

90.15 **If you wish, make the last digit tens and add what is before it as if it were units, and then cast out sevens. Then you make the remainder tens and you add it to what is before it as if it were units, and you cast out again.**

90.18 For example, in our preceding example we make the last five tens, and we add to it the eight that is before it as if it were units, giving fifty-eight. We cast out sevens, and the remainder is two. Then we make it tens and we add to it the zero before it. Since there is no number there, it gives twenty, whose remainder is six. We make it tens, and we add to it the six before it, giving sixty-six. Its remainder is three. We make it tens and we add to it the four that is before it, giving thirty-four. Its remainder is six, which is the answer.

91.1 **Subsection on the way to test [calculations] by casting-out.**

91.2 **For addition, you cast out each line, add their remainders, and cast it out. The remainder is the answer. Then you cast out the sum in the problem, which will agree with the answer.**

91.4 For example, we added forty-three, whose figure is 43, to sixty-four, whose figure is 64. The sum is one hundred seven, whose figure is 107. We can check this problem and other problems by casting out nines or eights or sevens. However, many people cast out sevens. We want to check this problem likewise, as well as the other examples. We add the one, the remainder of the addend, to the one, the remainder of the augend, giving two, which is the answer. If the sum of the two remainders can be cast out, we cast it out also, and its remainder is the answer. Then we cast out the sum, leaving two, which agrees with the answer. So know it.

91.12 **For subtraction, you cast out the minuend and you keep the remainder in mind. Then you cast out the subtrahend and you drop its residue from the remembered number. If it is smaller, add the modulus to it and drop it from the sum. The remainder is the answer. Then cast out the remainder in the problem. This should agree with the answer. Or, you add the residue of the subtrahend to the residue of the remainder. This should agree with the residue of the minuend.**

91.16 For example, we subtracted seventy-four, whose figure is 74, from ninety-six, whose figure is 96. The remainder is twenty-two, whose figure is 22. If we want to check it, we keep in mind the residual five of the minuend. Then we cast out the subtrahend, whose remainder is four. We drop it from the remembered five, leaving one, which is the answer.

92.3 And if this residual four of the subtrahend were greater than the residual five of the minuend, then we add [the modulus] to the five, the number of the minuend: nine if it is nine, eight if it is eight, or seven if it is seven. Then we cast out the remainder [in the problem], leaving one, which agrees with the answer. And if we add the four, the residue of the subtrahend, to the one, the residue of the remainder, it is the same as the five, the residue of the minuend. So know it.

92.8 **For multiplication, you cast out the two multiplicands and you multiply the remainder of one of them by the remainder of the other, and you cast it out. The remainder is the answer. Then you cast out the result of the multiplication, which should agree with the answer.**

92.10 For example, we multiply twelve, whose figure is 12, by sixteen, whose figure is 16. The result of the multiplication is one hundred ninety-two, whose figure is 192. We multiply the residual five of the multiplicand by the residual two of the multiplier, giving ten. Its remainder is three, which is the answer. Then we cast out the result of the multiplication, leaving three, which is equal to the answer.

92.15 **This generalizes to whole numbers and fractions after numerating them, *that is, when each of them in the problem becomes of a type [that is] one fraction.***

92.17 *For example, suppose someone said, "Multiply a third by fourteen and a fourth". The result of the multiplication is four and three fourths*, according to what will come in the work on fractions,[38] almighty God willing. *If we want to check it [by casting out sevens], we multiply the third, which is the remainder of one of the two multiplicands, by the fourth, the remainder of the other multiplicand, resulting in a third of a fourth. Its remainder after its numeration is one, which is a third of a fourth, which is the answer. Then we numerate the result of the multiplication, which gives nineteen fourths. Its remainder is five fourths. We numerate it by a third by multiplying it by three, giving fifteen. Its remainder is one, which is a third of a fourth, equal to the answer in quantity and quality.*[39]

93.6 **For division and denomination, you cast out the result and the divisor or denominating number. You multiply the remainder of one of them by the remainder of the other, and keep it in mind. The remainder is the answer. Then you cast out the dividend or denominated number, which should agree with the answer. This method also generalizes to whole numbers and to fractions, after numerating them.**

93.10 Here is an example of division. We divide one thousand four hundred eighty-eight, whose figure is 1488, by twelve. The result of the division is one hundred twenty-four, whose figure is 124. If we want to check it, we multiply the residual five of the result by the residual five of the divisor, giving twenty-five. Its remainder is four, which is the answer. Then we cast out the dividend, whose remainder is four, which agrees with the answer.

93.15 Here is an example with fractions. *Suppose someone said to you, "Divide five sixths and three fourths by a half". The result of the division is three and a sixth, and its remainder,*

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[38] I.e., the section on multiplying fractions, at 149.1.

[39] Copied from (Ibn al-Bannāʾ 1994, 248.11-17).

after numerating it, is five sixths. We multiply it by one, the numerator of the half, which is the divisor, giving five halves of a sixth. We then multiply it by two so it becomes fourths of sixths, agreeing in the numeration with the numerated dividend. It gives ten. Its remainder is three fourths of a sixth, which is the answer. Then we cast out the numerated dividend, which is thirty-eight fourths of a sixth. Its remainder is three fourths of a sixth, which is equal to the answer.[40]

94.1 Here is an example of denomination. Suppose someone said, "Denominate eleven with fifteen". The result of the denomination is three fifths and two thirds of a fifth. The remainder after numerating it is four. We multiply it by the one, the remainder of the denominating number, giving four, which is the answer. Then we cast out the denominated number. Its remainder is four, equal to the answer.

94.5 Here is an example with fractions. *Suppose someone said, "Denominate two sixths and two thirds of a sixth with five eighths and a third of an eighth". It results in two-thirds, and the remainder of its numerator is two. We multiply it by the two, the remainder from numerating the denominating number, giving four thirds of a third of an eighth. Then [we multiply it] by the six, giving twenty-four thirds of a third of a sixth of an eighth. Its remainder is three, which is the answer. Then we cast out the numeration of the numerator. Its remainder is one. We multiply it by the three, then by the eight. It gives twenty-four thirds of a third of a sixth of an eighth. Its remainder is three, which equals the answer.*

94.11 *It is necessary that everything in the problem be reduced to a finer fraction, which is the part named with all of the denominators. The meaning of numeration shall be clarified later, with the help of almighty God, may he be exalted.*[41]

95.1 **Section Four, on multiplication and understanding its subtleties.**

95.2 **Multiplication consists of the duplication of one of two numbers by however many units are in the other.**

95.3 *This section covers two types. [In one] type, in putting down the multiplier, each one of them is equal to the one of the multiplicand. Here the duplication clearly occurs in both the term and the meaning.*

95.6 *In the second type, all of what is in the multiplier in units is equal to the one of the multiplicand. So the [number of] units in the multiplier is the number of what is in one of the multiplicand in parts. This type is called conversion, and the duplication occurs in the term but not in the meaning.*

95.10 *Suppose someone said, "Three men: each of them has five dirhams". You multiply five by three, which gives fifteen dirhams. This is duplication in the term and in the meaning.*

95.12 *Suppose someone said, "Five dirhams: how many thirds does it contain?" You multiply five by three, which gives fifteen thirds. Here the duplication is only in the term, and as for the meaning, fifteen thirds are exactly five.*[42]

[40] Copied from (Ibn al-Bannāʾ 1994, 248.18-249.4).

[41] Copied from (Ibn al-Bannāʾ 1994, 249.5-11). Ibn al-Bannā gives a section on numeration in *Lifting the Veil* at p. 272.14. Al-Hawārī covers numeration beginning at 135.8.

[42] Copied from (Ibn al-Bannāʾ 1994, 255.3-13).

95.15 **[Multiplication] is divided into three kinds;** THAT IS, WITH REGARD TO PROCEDURE. **The first kind is by shifting, the second by half-shifting, and the third without any shifting.**

95.17 **The first kind, which is multiplication by shifting, calls for erasing, and is called sleeper [multiplication]. You write the multiplicand and the multiplier on two lines so that the first digit of the multiplier is below the last digit of the multiplicand. Then you multiply it by all the digits of the multiplier. You begin by writing the result in its place, traversing the line, continuing along the line of the multiplicand. Then you shift the number of the multiplier so that it is below the digit that follows those before it. Then you multiply it by all the digits of the lower number, like the first time. Whenever you multiply by a number, you add the result with what is over the head of that number from the previous result, and you put it where it belongs. This procedure is universal for all problems of multiplication.**

96.3 For example, if we want to multiply forty-three by fifty-four, we write down the forty-three, the multiplicand, on a line, and the fifty-four, the multiplier, on another line, so that the first digit of the multiplier is below the last digit of the multiplicand, as mentioned. Here is the figure:

	4	3
5	4	

We multiply the last digit[43] of the multiplicand, which is the four, by the five, the last digit of the multiplier, giving twenty. We put a zero above the five, and the twenty, in the form of a two, after the zero. We also multiply it by the four below it, giving sixteen. We put the six in its place, and the ten, in the form of a one, in place of the zero. These are included with the line of the multiplicand. Then we shift the multiplier back one place, so the four is below the three, and the five is below the six. This is the figure:

2	1	6	3
		5	4

96.13 Then we likewise multiply the shifted three below it[44] by the entire multiplier, and we add that to what is above it. We first multiply it by the five, giving fifteen. We add it to the sixteen above it, giving thirty-one. We put the one in place of the six, and the thirty, in the form of a three, in place of the ten. Then we likewise multiply it by the four, giving twelve. We put the two in its place, and we add the ten, in the form of a one, to the one that is in the second rank, giving two. We replace it with it, and this completes the work. So the result is two thousand three hundred twenty-two, and its figure is 2322.

97.1 **The other type is known as vertical [multiplication]. You set up the two multiplicands in two vertical lines so that the first digit of the multiplier is opposite the last digit**

[43] We translate as "last digit" what is more literally "what is in the last rank".
[44] This is misstated. The multiplier, not the three, is shifted.

of the multiplicand. You proceed in multiplying them just as you did in the sleeper method, by shifting and erasing.

97.4 For example, suppose we want to multiply forty-two by thirty-seven. We put down the multiplicand in a column, as mentioned, and the multiplier as well, so that its first digit is next to the last digit of the multiplicand, and this is the figure:

2	
4	7
	3

97.7 Then we multiply the last digit of the multiplicand, which is four, by all the digits of the multiplier, as before. We first multiply it by the three, giving twelve. We put the two adjacent to the three in the column of the multiplicand, as before, and the ten, in the form of a one, below the two. Then we likewise multiply it by the seven next to it, giving twenty-eight. We put the eight in its place, and we add the twenty, in the form of a two, to the two that is below that rank, giving four. We replace it with it. Then we shift the multiplier back one place, so the seven is next to the two, and the three is next to the eight, as in this figure:

2	7
8	3
4	
1	

97.14 Next we likewise multiply the two by the whole multiplier, as before. We first multiply it by the three, giving six. We add it to the eight next to it, giving fourteen. We put the four in place of the eight, and we add the ten, in the form of a one, to the four in the rank below it, giving five. We replace it with it. Then we likewise multiply the two by the seven, giving fourteen. We put the four in its place, and we add the ten, in the form of a one, to the four that is in the rank below it, giving five. We replace it with it. This completes the work. The result is one thousand five hundred fifty-four, whose figure is 1554.

98.4 **The second kind is multiplication by half-shifting, which only works for two equal numbers. In this scheme you write down one of the equal numbers on a line, and you put marks in the form of dots between its digits. Then you multiply the last digit by itself and you put the result above it. Then you double it and you shift it by writing it down in place of the dot preceding it. Then you multiply the preceding digit by the shifted number and by itself, and you write the results of each multiplication above them. Then you double the digit that you just multiplied, as you did before. Then you shift it in place of the dot that precedes it. Then you shift the first doubled number, [adding] to its calculation. Then you multiply the digit preceding the dot that was replaced by all the doubled number, then by itself, as you did before. You continue**

in the same way, doubling, shifting, and multiplying until you reach the end of the line.[45]

98.15 For example, if we want to multiply four hundred sixty-three by itself, we put it down on a line and we separate the numbers with dots as mentioned, like in this figure: 4 • 6 • 3. We multiply the last four by itself, giving sixteen. We put the six above the four, and the ten, in the form of a one, after the six. Then we double it, giving eight, and we put it in place of the dot preceding it. So it yields this figure:

1	6				
	4	8	6	•	3

98.20 Then we multiply the preceding six facing it by the shifted eight, giving forty-eight. We put the eight above the eight of the multiplier, and we add the forty, in the form of a four, to the six in the next rank, giving ten. We put a zero in its place and we add the ten, in the form of a one, to the one in the next place, giving two. We replace it with it. Then we likewise multiply the six by itself, giving thirty-six. We put this six above it, and we add the thirty, in the form of a three, to the eight that is in the next rank, giving eleven. We put the one in place of the eight, and the ten, in the form of a one, in place of the zero. We then move the six back, also doubling, giving twelve. We put the two in place of the dot before it, and we add the eight to the ten, which resulted from doubling [the six], in the form of a one, giving nine. We put it down in place of the six, as in this figure:

2	1	1	6		
	4	8	9	2	3

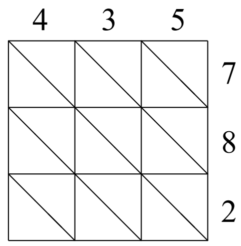
99.7 Then we likewise multiply the preceding three facing it by the whole shifted number and by itself, as before. We multiply it first by the nine, giving twenty-seven. We add to it the sixteen above it, giving forty-three. We put the three in place of the six which is above it, giving three, and the forty, in the form of a four, in place of the ten. Then we also multiply it by the two, giving six. We put it above it. Then we also multiply it by itself, giving nine. We put it above it. And this completes the work. The result is two hundred fourteen thousand three hundred sixty-nine, and its figure is 214369. So understand it and pursue similar problems the same way. Proceed by the power of almighty God.

99.14 **The third kind is multiplication without shifting, of which there are several types. One is table multiplication, in which you draw a quadrilateral surface, extending length and width according to the ranks in the two numbers to be multiplied. You draw diagonals through its squares from the lower right to the upper left, and you write the multiplicand above the quadrilateral, matching each digit with a column. Then you write the multiplier down the left or right side of the quadrilateral, so that each digit also corresponds to a row. Then you multiply digit after digit of the multiplicand by all the digits of the multiplier, and you put the digits of each rank in the intersecting square.** And the meaning of "intersection" is where they meet. **You put**

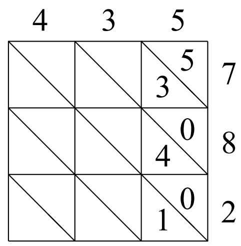
[45] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 15.2-9).

the units above the diagonal and the tens below it. Then you begin adding from the upper right corner. You add what is between the diagonals, without erasing, and you write each number in its rank, and you add the tens of each sum to the next diagonal. You then put it together, and the sum that you obtain is the result.

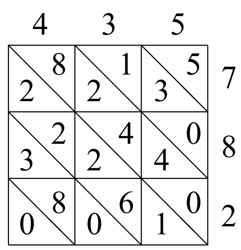
100.5 For example, suppose we want to multiply four hundred thirty-five by two hundred eighty-seven. We draw the quadrilateral as he mentioned, with the multiplier on the left or right of the quadrilateral, and the multiplicand above the quadrilateral. Each number is above a column of the quadrilateral, and similarly we write the multiplier on the right of the quadrilateral, as in this figure:



100.9 We multiply the five by the seven, giving thirty-five. We put down the five above the diagonal in the corresponding square, and the thirty, in the form of a three, below it. Then we likewise multiply the five by the eight, giving forty. We put the zero above the diagonal of the corresponding square, and the forty, in the form of a four, below it. Then we likewise multiply the five by the two, giving ten. We put the zero above the corresponding square, and the ten, in the form of a one, below it. This is the figure:



101.1 Then we do the same for the three. We multiply it by all the digits of the multiplier and we put the results of each of them in the corresponding square as before, and we do likewise for the remaining four of the multiplicand. Then we finish the multiplication. The numbers are situated in the quadrilateral in its entirety as in this figure:



101.5 Then we begin adding. We raise the five that is above the first diagonal in the upper right corner, as mentioned. Then we likewise add what is between the first and second diagonals, which are three and one, which add to four. We raise it after the five that we raised first. Then we likewise add what is between the second and third diagonals, which are four and four and two and eight. They add to eighteen. So we raise the eight after the raised four, and we add the ten, in the form of a one, with what is between the third and fourth diagonals, which are one and six and two and two and two. They add to fourteen. So we raise the four after the raised eight, and we add the ten, in the form of a one, with what is between the fourth and fifth diagonals, which are eight and three, giving twelve. We raise the two after the raised four, and the ten, in the form of a one, after it. This completes the work. All of this together is the required number, which is one hundred twenty-four thousand eight hundred forty-five, and its figure is 124845.

101.16 **Another [type] is vertical multiplication, in which you draw two vertical lines with some empty space between them, and you draw the two multiplicands beside them. Then you multiply one of them, digit by digit, by all the digits of the other, and you put the results in the space between the lines, taking into account the ranks of the indexes.**

102.1 For example, suppose we want to multiply one hundred eighty-three by three hundred forty-seven. We put down the multiplicand vertically in a line, and the multiplier parallel to the multiplicand, and a line after the multiplicand with another line before the multiplier as mentioned, as in this figure:

7		3
4		8
3		1

102.4 We multiply the three, which is the first [digit] of the multiplicand, by all the digits of the multiplier. We multiply it first by the seven, giving twenty-one. Then we put the one down next to the three beside the line, and the twenty, in the form of a two, below it. Then we multiply it likewise by the four, giving twelve. We add to it the two that is in the second rank of the result, giving fourteen. We put the four down next to the two, and the ten, in the form of a one, below it. Then we multiply it likewise by the three, giving nine, and we add it to the one that is in the third rank of the result, giving ten. We put the zero down next to the one, and the ten, in the form of a one, below it. This is the figure:

7		1	3
4	4	2	8
3	0	1	1
		1	

102.11 Then we similarly multiply the eight, which is in the second rank of the multiplicand, by all the digits of the multiplier. We multiply it first by the seven, giving fifty-six. We add it to the four that is in the second rank of the result, giving sixty. We put the zero down next to the four, and the sixty, in the form of a six, next to the zero that is after the second

[rank]. Then we multiply it also by the four, giving thirty-two. We add it to the six that is in the third rank of the result, giving thirty-eight. We put the eight down next to it, and we add the thirty, in the form of a three, to the one that is after that rank, giving four. We put it down next to it. Then we multiply it also by the three, giving twenty-four. We add it to the four that is in the fourth rank of the result, giving twenty-eight. We put the eight down next to it, and the twenty, in the form of a two, after that digit.

103.3 Then we multiply it also by the one, which is in the third rank of the multiplicand, again by all of the multiplier. We multiply it first by the seven, giving seven. We add it to the eight that is in the third rank of the result, giving fifteen. We put the five next to it, and we add the ten, in the form of a one, to the eight that is after that rank, giving nine. We put it down next to it. Then we multiply it also by the four, giving four. We add it to the nine that is in the fourth rank of the result, giving thirteen. We put the three down next to it, and we add the ten, in the form of a one, to the two that is after that rank, giving three. We put it down next to it. Then we multiply it also by the three, giving three. We add it to the three that is in the fifth rank of the result, giving six. We put it down next to it. This completes the work, and this is the figure:

7					1	3
4			0	4	2	8
3	5	8	6	0	1	1
	3	9	8	4	1	
			6	3	2	

103.12 The result is sixty-three thousand five hundred one. This is the vertical line adjacent to the multiplier. So know it.

103.14 **[Another type] is sleeper multiplication. You put the two multiplicands in two parallel lines. Then you multiply each digit of one of them by each digit of the other, and you put down the result taking into account the ranks of the indexes. You may start the multiplication from the first digit or the last. This type is also called multiplication by indexes.**

104.1 I say that we need here an introductory remark for this type that may also be useful for *rūmī* multiplication. The result of multiplying the units is units, since it comes from multiplying a number by a number. The index of the result is the sum of the indexes of the two multiplied numbers less one, as shown by the author, God be satisfied by him, in the section on addition in *Lifting the Veil*.[46] Certainly the index of the multiplicand is one and the index of the multiplier is one, so their sum is two. Dropping one leaves one, and the name of the one, according to what [was said] before, is units.[47] So the result of multiplying the units by the units is units. Likewise, their product by the tens is also tens, and by the hundreds is hundreds. And multiplying the tens by the tens is hundreds, and by the hundreds is thousands. And the hundreds by the hundreds is tens of thousands. This is the end of the introductory remark.

[46] The reference is to (Ibn al-Bannāʾ 1994, 215.12).
[47] Above at 71.6.

104.10 Suppose we want to multiply two hundred fifty-three by nine hundred eighty-seven. We put them down in two parallel lines, as mentioned. This is the figure:

2	5	3
9	8	7

104.12 We multiply the three, which is the units digit of the multiplicand, by all the ranks of the multiplier. We multiply it first by the seven, giving twenty-one. We put down the one in the units rank, as before, and the twenty, in the form of a two, after it. Then we multiply it also by the eight, giving twenty-four. We add it to the two that is in the tens rank of the result, since they are of the same species, as [mentioned] before, giving twenty-six. We put the six above the two, and the twenty, in the form of a two, after it. Then we multiply it also by the nine, giving twenty-seven. We add to it the two that is in the hundreds rank of the result, because they are also of the same species, giving twenty-nine. We put the nine above the two, and the twenty, in the form of a two, after it. This is the figure:

	9	6	
2	2	2	1
	2	5	3
	9	8	7

105.1 We likewise multiply the five that is in the tens rank of the multiplicand by all the ranks of the multiplier. We multiply it first by the seven, giving thirty-five. We add to it the six that is in the tens rank of the result, since it is also of the same species, giving forty-one. We put the one above the six, and we add the forty, in the form of a four, to the nine that is after that rank, giving thirteen. We put the three above the nine, and we add the ten, in the form of a one, to the two that is after that rank, giving three. We put it down above the two. Then we multiply it also by the eight, giving forty. We add to it the three that is in the hundreds rank of the result, giving forty-three. We put the three in its place[48] and we add the forty, in the form of a four, to the three that is after that rank, giving seven. We put it above the three. Then we multiply it also by the nine, giving forty-five. We add to it the seven that is in the thousands rank of the result, giving fifty-two. We put the two above the seven, and the fifty, in the form of a five, after it. This is the figure:

	2			
	7	3	1	
	3	9	6	
5	2	2	2	1
		2	5	3
		9	8	7

105.14 Similarly we multiply the two that is in the hundreds rank of the multiplicand by all the ranks of the multiplier. We multiply it first by the seven, giving fourteen. We add it to the

[48] The three is already there, so there is no need to write it.

three that is in the hundreds rank of the result, since they are also of the same species, giving seventeen. We put the seven above the three, and we add the ten, in the form of a one, to the two that is after that rank, giving three. We put it above the two. Then we multiply it also by the eight, giving sixteen. We add it to the three that is in the thousands rank of the result, giving nineteen. We put the nine above the three, and we add the ten, in the form of a one, to the five that is after that rank, giving six. We put it above the five. Then we multiply it also by the nine, giving eighteen. We add it to the six that is in the ten thousands rank of the result, giving twenty-four. We put the four above the six, and the twenty, in the form of a two, after it. This completes the work, and its figure is:

		9			
		3			
		2	7		
	4	7	3	1	
	6	3	9	6	
2	5	2	2	2	1
			2	5	3
			9	8	7

106.8 The result of the multiplication is two hundred forty-nine thousand seven hundred eleven, which is the required number. And if we wish, we can follow the method by starting from the last places. So know it.

106.11 **Another type [of multiplication] requires that the [numbers of] digits of the multiplicands be equal, and that the digits in each rank of the ranks in each line also be equal. The way to write it down is similar to the way [for the method] by erasing. Then you write a one below the first rank of the ranks of the upper line, and a two below the second [rank], likewise increasing by one until you reach the last rank of the multiplicand. What is below it is shared with the first rank of the multiplier.** From the second rank of the multiplier you begin decreasing one by one until you reach the last rank of the multiplier. **These written numbers are all on a third line. The indexes of the ranks of the multiplicand are in the correct order, but the indexes of the ranks of the multiplier are reversed. Then you multiply the number of the rank of the multiplicand by the number of the rank of the multiplier. The result is multiplied by what is in the written line, and the result is the required number. This type of multiplication is called "by repetition".**

107.6 For example, suppose someone said, "Multiply four hundred forty-four by three hundred thirty-three". We put them down on two lines, as mentioned. This gives the figure:

		4	4	4
3	3	3		

Then we write a one below the first four, a two below the second, and a three below the third. This is the first rank of the multiplier. Then we write a two below the second three, since from this point we begin decreasing one by one, as mentioned, and a one below the third three, as in this figure:

$$\begin{array}{ccccc} & & 4 & 4 & 4 \\ 3 & 3 & 3 & & \\ \hline 1 & 2 & 3 & 2 & 1 \end{array}$$

As mentioned, the indexes of the ranks of the multiplicand are in the correct order, and the indexes of the ranks of the multiplier are reversed, since the index of the three in the first [rank] of the multiplier is one, and we wrote it below the third three, which is its last rank.

107.15 Then we multiply the number of a rank of the multiplicand, which in this example is four, by the number of a rank of the multiplier, which in this example is three, giving twelve. We multiply it by the new line. We multiply it first by the one, which is the last in the line, giving twelve. We put the two above it, and the ten, in the form of a one, after it. Then we multiply it also by the two in the fourth [position] of the line, giving twenty-four. We put the four above the two, and we add the twenty, in the form of a two, to the two after that rank, giving four. We put it above it. Then we multiply it also by the three in the third [position] of the line, giving thirty-six. We put the six above the three, and we add the thirty, in the form of a three, to the four that is after that rank, giving seven. We put it above it. Then we multiply it also by the two in the second [position] of the line, giving twenty-four. We put the four above the two, and we add the twenty, in the form of a two, to the six that is after that rank, giving eight. We put it above it. Then we multiply it also by the one in the first [position] of the line, giving twelve. We put the two above it, and we add the ten, in the form of a one, to the four that is after that rank, giving five. We put it above it. That completes the work, and this is the figure:

$$\begin{array}{cccccc} & 4 & 7 & 8 & 5 & \\ 1 & 2 & 4 & 6 & 4 & 2 \\ \hline & & & 4 & 4 & 4 \\ & 3 & 3 & 3 & & \\ \cline{2-6} & 1 & 2 & 3 & 2 & 1 \end{array}$$

108.7 The result of the multiplication is one hundred forty-seven thousand eight hundred fifty-two, which is the required number.

108.9 **Another [type] is multiplication by excess, in which you denominate the excess over ten of one of the two multiplicands with the ten. Then you take that ratio of the other [multiplicand]. You add it to it and you make it tens. If the ratio has a fractional part, you take it with respect to the ten[49] and you put it in the units place.**

108.13 For example, suppose we want to multiply twelve by fifteen. We denominate the two, the excess over the ten in the multiplicand, with the ten, yielding a fifth. We take a fifth of the fifteen, the multiplier, giving three. We add it to it, giving eighteen, and we make it tens.

[49] I.e., divide it by 10.

So it is one hundred eighty. This is the result, which is the required product, and its figure is 180. And if we denominate the five, the excess over the ten in the multiplier, with the ten, it yields a half. We take half of the twelve, giving six. We add it to it, again getting eighteen. We make it tens to get one hundred eighty, which is the result, as before.

109.1 Another example: suppose we want to multiply thirteen by seventeen. We denominate the three, the excess over the ten in the multiplicand, with the ten, yielding three tenths. We take three tenths of seventeen, the multiplier, giving five and a tenth. We add it to it, giving twenty-two and a tenth. We make it tens, and we take the fraction of the ten, which is a tenth, giving one. This we put it in the units place, as mentioned. That yields two hundred twenty-one, which is the required number, and its figure is 221.

109.7 **Another type is known as denomination. Here you add the two multiplicands, then you denominate one of them with the sum. Then you take that ratio of the other [multiplicand] and you multiply it by the sum, so the result is the required number.**[50]

109.10 For example, suppose we want to multiply six by twelve. We add them, giving eighteen. Then we denominate one of them with it. We denominate the six with it, giving a third. We take a third of the multiplier, which is the twelve, giving four. We multiply it by the sum. The result is the required number, which is seventy-two, and its figure is 72. And if we denominate the twelve with the sum, we get two-thirds. So we take two thirds of the six, giving four. We multiply it by the sum, and the result is the required number, as before.

109.16 And recall what the jurist Abū Muḥammad ibn Ḥajjāj, known as Ibn al-Yāsamīn, said about this method: "if we denominate one of them with the sum, then we drop that ratio of the number from itself and we multiply the remainder by the sum, it results in the required number".

109.19 **Another type is also known as denomination. You denominate the most convenient of the two multiplicands with whatever simple power of ten you wish, or you divide it by it. Then you multiply the result of the denomination or division by the other [multiplicand]. Then you raise each [digit] of the result by the power of ten of the divisor. The outcome is the required number.**

110.1 **If the result of dividing or denominating gives a whole number only by adding something to it or subtracting it from it, then do it. Then you multiply the added quantity by the number you did not add it to, and you subtract the outcome from the result. If you worked it by subtraction, then add the outcome to the result.**[51]

110.5 And by "simple power of ten" we mean that the first [non-zero] rank is equal to the ten or the hundred or the like.

110.6 Example problem: we want to multiply twenty-four by eight. So we denominate the eight, the multiplier, with whatever simple power of ten we wish. Supposing it is ten, it gives four-fifths. We multiply it by the twenty-four, resulting in nineteen and a fifth. We raise each [digit] by ten, the denominated power of ten, giving one hundred ninety, and a fifth of one [raised by] ten gives two. We add it to it, so the outcome is the required number, and that is one hundred ninety-two, and its figure is 192.

[50] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 16.5-7).
[51] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 16.8-13).

110.12 And another example: suppose we want to multiply twelve by fifteen. We divide the fifteen by the ten, resulting in one and a half. We multiply it by the twelve. The result of the multiplication is eighteen. We raise each [digit] by the ten, the divided power of ten, yielding one hundred eighty, which is the required number.

110.16 And if we wish, we can instead drop the five from the fifteen and divide the remaining ten by the ten, resulting in one. We multiply it by the twelve, giving twelve. We make it tens and we add to it the product of the subtracted five by the twelve, since it was not subtracted from it. The sum is one hundred eighty, which is the required number, as before.

111.1 Another example: we want to multiply three by fifteen. So we add two to the three, giving five. We denominate it with the ten, giving a half. We multiply it by the fifteen. The result of the multiplication is seven and a half. We raise each [digit] by ten, the power of ten, and half of the ten, to get seventy-five. We drop from it the product of the added two by the fifteen, since it was not added to it. The remainder is the required number, and that is forty-five, and its figure is 45.

111.7 **[Another type] is multiplication of "nines", with the requirements that the [numbers of] ranks in the two lines be equal, one of them consist of all nines, and the digits of the other be equal. A description of the procedure is that you write down the two parallel lines, one of them below the other, and you put dots above them, as many as there are places in them. You multiply the digit of the place of one of them by the digit of the place of the other. You put the units of the result in the first of the dots, and its tens in the middle of the remaining dots. You note the difference between the nine and the digit of the multiplier. Then with [this difference] you fill in what is between the two digits of the results, I mean the units and the tens, and you fill the remaining dots with the digit that is different from the nine. What this yields is the answer.**[52]

111.15 For example, we want to multiply four hundred forty-four by nine hundred ninety-nine. We put them down in two parallel lines, as mentioned, and we put dots above them, as many dots as there are places in both of them, and that is six. This is the figure:

•	•	•	•	•	•
			4	4	4
			9	9	9

111.18 We multiply the number of a rank of the multiplicand by the number of a rank of the multiplier, which is nine by four, giving thirty-six. We put down the six on the first of the dots, and five [dots] remain. We put down the thirty, in the form of a three, on the third of the five [remaining dots], since it is in the middle. Then we take the difference between the nine and the four, which is five. We fill in with it both of the dots between the units, which is six, and the tens, which is three, and we fill in the remaining dots with the four. This is what he meant when he said "with the number that is different from the nine". This is the figure:

[52] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 16.14-17.2).

4	4	3	5	5	6
			4	4	4
			9	9	9

112.3 The result is the number that is above the two multiplicands, and that is four hundred forty-three thousand five hundred fifty-six. So know it, and solve similar examples the same way.

112.5 **Another type of multiplication of nines has no condition. Instead, the digits in one of the lines are nines, and the digits in the other line can be whatever they are, and the number of places can also be whatever they are. To work it out, you add as many zeros to the ranks of the other line as the number of ranks of the nines. Then you subtract from the total the number different from the nines, leaving the answer.**[53]

112.10 For example, we want to multiply nine hundred ninety-nine by nine thousand three hundred fifty-four. We add to the multiplier as many zeros as the number of ranks of the nines, which is three. So it comes to nine million three hundred fifty-four thousand, whose figure is 9354000. Then we drop the number that is not nines from it, which is the multiplier. The remainder is the required number, which is nine million three hundred forty-four thousand six hundred forty-six. Its figure is 9344646.

112.16 **Another type is known as squaring. You take half of the sum of the multiplicands and you square it. You subtract from the result a square of half of the difference between them. The remainder is the result of the multiplication.**[54]

113.1 For example, we want to multiply seventeen by nineteen. We take half of their sum, which is eighteen. We multiply it by itself, which is the meaning of "squaring", as [mentioned] before, giving three hundred twenty-four. We drop from it one, which is a square of half of the difference between the multiplicands, leaving three hundred twenty-three, which is the required number, and its figure is 323.

113.6 **Another type is also known as squaring, which is that you multiply a square of one of the two multiplicands by what results from the ratio of the other to the number you squared, or you divide a square of one of them by the result of dividing the squared one by the other one.**

113.9 For example, we want to multiply twenty-five by fifteen. So we multiply a square of twenty-five, which is six hundred twenty-five, by three fifths, which is the ratio of the fifteen to the twenty-five. The result is the required number, and that is three hundred seventy-five, and its figure is 375. Or we divide a square of the fifteen, which is two hundred twenty-five, by the three fifths, which is the result of dividing the fifteen that we squared by the twenty-five. The result is the required number, like before. So know it.

113.16 **Another type is that you multiply the difference between the two multiplicands by the greater of them, and you drop the result from a square of the greater. Or you**

[53] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.3-7).
[54] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.8-10).

multiply the difference by the smaller of them and you add the result to a square of the smaller. The result is the required number.[55]

113.19 For example, suppose someone said, "Multiply thirty-six by fourteen". We multiply the twenty-two, which is the difference between the multiplicands, by thirty-six, the greater of the multiplicands, resulting in seven hundred ninety-two. We drop that from a square of the greater, which is one thousand two hundred ninety-six. The remainder is the required number, which is five hundred four, and its figure is 504. Or we multiply the twenty-two, the difference, by the fourteen, the smaller multiplicand, resulting in three hundred eight. We add it to a square of the smaller, which is one hundred ninety-six. The sum is the required number, like before. So know it.

114.4 **And if you multiply a number with zeros by a number with zeros, multiply the parts of one by the parts of the other, stripped of the zeros. Then you dress the result with all the zeros. The outcome is the required number,**[56] SINCE MULTIPLYING THE NUMBER BY THE ZERO OR THE ZERO BY THE NUMBER IS IDENTICAL. IT COMES FROM VOIDING THE NUMBER OR DUPLICATING ZERO. NEITHER OF THESE GIVES A NUMBER, SO ITS SIGN IS ALWAYS A ZERO.

114.8 For example, suppose someone said, "Multiply thirty by one hundred forty". We remove the zero from the thirty and we set it aside, leaving three. Likewise we remove it from the one hundred forty and we set it aside, leaving fourteen. We multiply it by the three and we add the two zeros that were set aside to the result. The outcome is the required number, which is four thousand two hundred, and its figure is 4200.

114.12 **The upper limit on the [number of] ranks of the result is the sum of the ranks of the two multiplicands**, since the greatest [number] one can have in a single rank is a nine, and multiplying nine by nine is eighty-one. Thus multiplication in the units rank can give tens.

114.15 **To check it, you divide the result by one of the multiplicands. The result is the other [multiplicand].** For example, in the previous example we divide the eighty-one by the nine, which is one of the multiplicands. The result is nine, which is the other multiplicand.

115.1 **The student should memorize the following breakdown and master it, which is:**

115.2 **Multiplying a number by one or multiplying one by it leaves the number unduplicated,** like two by one gives two, or one by five gives five.

115.5 **And two by two gives four, and by each number that follows gives an additional two.** For successive numbers, multiplying two by three gives six, and by four gives eight, and by five gives ten, and so on up to the ten, since this is the traditional limit of the breakdown.

115.9 **And three by three gives nine, and by each number that follows gives an additional three.** He means "for successive numbers" here, too. Thus multiplying it by four gives twelve, and by five gives fifteen, and so on up to ten.

[55] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.11-13).
[56] Copied from Ibn al-Yāsamīn, (Zemouli n.d., 17.13-15).

115.12 **And four by four gives sixteen, and by each number that follows gives an additional four. And five by five gives twenty-five, and by each number that follows gives an additional five. And six by six gives thirty-six, and by each number that follows gives an additional six. And seven by seven gives forty-nine, and by each number that follows gives an additional seven. And eight by eight gives sixty-four, and by each number that follows gives an additional eight.** And for each of them for successive [numbers], just as the ones before. **And nine by nine gives eighty-one, and by ten gives ninety. And ten by ten gives a hundred.** So know it. Proceed by the power of almighty God.

117.1 **Section Five, on division.**

117.2 **Division is the decomposition of the dividend into equal parts in such a way that its number is equal to what is in the divisor in units.**[57] *This applies to discrete quantities.*[58]

117.5 **Division is also viewed as the ratio of one of two numbers with respect to the other.**[59] *And this concerns continuous quantities.*[60]

117.7 **Most people view division in all circumstances as knowing how many whole units of the divisor are in the dividend.**

117.9 ***Division has two meanings. One of them is described first, which concerns the division of a type by another type, like dirhams by men. The other is described second, and concerns the division of a type by the same type. So the word "division" has two meanings. And it should not be given a single description without regard to circumstances, as most people do. The meaning many people give it is specifically the first meaning, and they ignore the second meaning. Their description without regard to circumstances misleads the general public to lump the two meanings together, or to think that division truly has one meaning, and this is not so.***[61]

117.16 *Here is an example of division with the first meaning. Divide fifteen dirhams among three men. We decompose the fifteen into three equal parts, which is how many units are in the divisor. So each part consists of five dirhams, which is how many whole units there are of that three, the divisor.*

118.1 *Here is an example of the second meaning. Divide a piece of wood of fifteen spans by a piece of wood of three spans. The intention here is, how many copies of the divisor are in the dividend? So we cut up the dividend into copies of the divisor. There are five parts in the dividend* of copies of the divisor*, each part equal to the divisor.*

118.6 *So the result of working out the division in each of the two meanings is five. But the units of the result of five in the first meaning is different from the units of the result of five in the second meaning, since in the first meaning it is the number that is in a part of the parts of the dividend in units, and in the second meaning it is the number of parts in the dividend. So, in the first meaning the dividend is divided into a given number of parts, and*

[57] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 119.13-14).
[58] Copied from (Ibn al-Bannāʾ 1994, 263.3).
[59] Copied from Ibn al-Yāsamīn, (Zemouli 1993, 119.16).
[60] Copied from (Ibn al-Bannāʾ 1994, 263.4).
[61] Copied from (Ibn al-Bannāʾ 1994, 263.4-9).

what is in each of these parts comes to be known through the division. And, in the second meaning, what is in each of the parts of the dividend is given in units, and the number of parts into which it is divided is what comes to be known through the division. Thus the second meaning is the opposite of the first meaning.[62] So know it.

118.14 **Division comes in two types: dividing a greater number by a smaller number and dividing a smaller number by a greater number. Dividing the smaller by the greater specifically is called "denomination"**, AND THE WORD "DIVISION" IS RESERVED FOR DIVIDING THE GREATER BY THE SMALLER.

118.17 **The usual way of dividing the greater by the smaller is that you write the dividend on a line, and you write the divisor below it, making sure that the greater is not below the smaller. Then look for a number to put below the first digit of the digits of the divisor so that when you multiply it by all of its digits, it either exhausts the entire dividend or it leaves a remainder smaller than the divisor, in which case you denominate it with it.**

119.1 For example, suppose we want to divide two hundred forty-five by twelve. We put the dividend down on a line, and the divisor twelve on a line below it, with the two below the four, and the ten, in the form of a one, below the two, as in this figure:

2	4	5
1	2	

So, it happens that the divisor is below the twenty-four. If it were smaller, say for example eleven or ten or something similar, we would still put down the last digit of the divisor below the last digit of the dividend.

119.7 In our example, we look for a number to put below the two of the divisor, since it is the first digit. We multiply it by all of it, so that it either exhausts the twenty-four above it, or it leaves a remainder less than the twelve. We find that it is two. We multiply this two first by the one that is in the last rank of the divisor, giving two. This exhausts the two above it. Then we likewise multiply it by the two below it, giving four, and it exhausts the four above it. Then we shift the twelve back one place below the five, and we look for a number to put below the two of the divisor so that when we multiply them, it exhausts what is above it. We find nothing, so we put a zero in its place, and five remains from the twelve. We denominate it with it, to get two sixths and half a sixth. We add it to the whole number, which gives the required number, and that is twenty and two sixths and half a sixth, and its figure is $\frac{1\ 2}{2\ 6}$ 20. Similar problems are worked out the same way.

119.18 **If you want, you can divide the dividend into parts and add the quotients to get the result. Or you can decompose the divisor into the numbers of which it is composed and make them denominators, and divide the dividend by them. Or you can reconcile the dividend and the divisor and divide the reconciled divisor by the reconciled dividend.**

[62] Copied from (Ibn al-Bannāʾ 1994, 263.10-264.8).

120.5 An example of the first: suppose we want to divide forty-four equally among eleven men. We split the forty-four into twenty-two and twenty-two. If we wish, we can partition it differently. So we divide the twenty-two by eleven, resulting in two, and similarly for the other division. We add them, giving four, which is the result of dividing forty-four by eleven.

120.10 An example of the second: suppose someone said, “Divide ninety-six by twelve”. We decompose the divisor twelve into what it is composed of, which is six and two, or three and four, which is similar. Suppose we decompose it the first way. We divide the ninety-six by the two first, resulting in forty-eight. We likewise divide it by the six, the other denominator, resulting in eight, which is the required number. And if we divide the dividend first by the six, then what results by the two, it still gives eight. So know it.

120.16 An example of the third: suppose someone said, “Divide thirty-five by fifteen”. You rework each of them by a fifth, which reconciles them. For the dividend, its fifth is seven, and for the divisor, its fifth is three. We divide seven by three, resulting in two and a third, which is equal to the result of dividing the thirty-five by the fifteen. So know it.

120.20 **[Another] type of division is specifically called apportionment. The way you work it out is that you add the apportioned parts and you take it to be a denominator. Then you multiply each of the apportioned parts by the dividend, and you divide the results by the denominator, giving the required number.**

121.4 For example, a man is bankrupt. Some people give him ten dinars, which he accepts. Suppose there are three donors, and they divide it according to their own wealth. One of them has four dinars, the second five, and the third six. We add up these parts, which surpass them, that is, the apportioned parts, and their sum is fifteen dinars. We make it a denominator. Then we multiply what the first has in hand by the dividend, ten, giving forty. We divide it by the denominator, resulting in two dinars and two-thirds of a dinar, which is his obligation of the ten. Likewise we multiply what the second has in hand by the ten, giving fifty, and we divide it by the denominator, resulting in three dinars and a third of a dinar, which is his obligation of the ten. Then we likewise multiply what the third has in hand by the ten, giving sixty, and we divide it by the denominator, resulting in four dinars, which is his obligation of the ten. So if we add up these three results it gives ten.

121.14 There are four other ways to do this. One of them is that we denominate what each of them has in hand with the denominator, and we multiply the result by the dividend. This gives the required number. The second is that we denominate the dividend with the denominator, and what results is called the part of the share. We multiply it by what each one has in hand, giving the required number. In the third, we divide the denominator by what each one has in hand, and we divide the dividend by the results, giving the required number. In the fourth we divide the denominator by the dividend and we divide what each one has in hand by the results. This gives the required number.

121.23 There are other ways [of solving this problem] which involve combining the proportion, switching it, or some of the other conditions of the proportion, as will be seen later, almighty God willing.[63] What we have described is sufficient for anyone to understand.

[63] Manipulations of proportions are covered starting at 196.16.

122.2 **In the case that the apportioned parts are fractions, multiply everything in the problem by the smallest number divisible by the denominators. And if all of the parts have a common divisor, remove it by exchanging the parts or reconciling them.**

122.5 For example: a man is bankrupt, and he receives twelve dinars. He has three donors, the first of whom has four dinars and a third of a dinar. The second has five dinars and a fourth of a dinar, and the third has six dinars and a sixth of a dinar. We multiply each of the parts that each of them has in hand by the smallest number divisible by their denominators. Knowing the smallest number divisible by the denominators in this example and in others is reached by decomposition.

122.10 *We decompose each of the denominators into the numbers of which it is composed, and we drop numbers from the second that are repeats of numbers in the first, and from numbers of the third that are repeats of numbers in the first and remaining in the second, and from numbers in the fourth that are repeats in the previous numbers, and so on to the last. Then we compose the remaining numbers by multiplication. If there is no repeated number, then they are all different, so multiplying them together gives the smallest number divisible by the denominators.*[64]

122.15 If we want to apply this in the present example, we find that the denominator of a third, which is three, cannot be decomposed. The denominator of a fourth, which is four, decomposes into two and two, and there is no repetition, so we keep them with the three. The denominator of a sixth decomposes into three and two. Both are repeats, so we discard them and we compose the rest, and that is three by two, giving six, and six by two, giving twelve, which is the least number divisible by the denominators.

122.19 We multiply everything in the problem by it. We multiply it first by what the first has in hand, resulting in fifty-two. We exchange it with what he has in hand. Then we likewise multiply it by what the second has in hand, resulting in sixty-three, and we also exchange it with what he has. Then we likewise multiply it by what the third has in hand, resulting in seventy-four, and we also exchange it with what he has in hand. Then we see if these parts have a common divisor, and we remove it and replace each one with its reconciled number, as mentioned.

123.3 To know how to remove common divisors in this example and others, *we decompose the numbers according to how they are composed, and we drop the repeated numbers in each of them from the total. Then we compose what is left of each of them by multiplication to get the reconciled amounts. Whenever no number remains, exchange it with one, since multiplying a number that has vanished involves no duplication. Thus the exchanged one is the reconciled amount. The common divisor of these numbers can always be obtained by means of the denominated part of the repeated number that was dropped.*[65]

123.9 If it is two, the common divisor is a half, and if it is five, then it is a fifth, and if it is ten, then it is a tenth, and if it is eleven, then it is a part of eleven, and so on.

123.11 After testing the parts mentioned in the example above by this procedure, we find them to be different, so we add them. This is the denominator, and it is one hundred eighty-

[64] Copied from (Ibn al-Bannāʾ 1994, 266.15-20).
[65] Copied from (Ibn al-Bannāʾ 1994, 267.1-5).

nine. So we multiply what each of them has in hand by the twelve and we divide by the denominator. The result for the first [donor] is three dinars and two ninths of a dinar and five sevenths of a ninth of a dinar, which is his obligation of the twelve. For the second it is four dinars, which is his obligation of the twelve. And for the third it is four dinars and six ninths of a dinar and two sevenths of a ninth of a dinar, which is his obligation of the twelve dinars. If we add these, they likewise result in twelve. And, if we wish, we can also work this out by the other methods mentioned above.

123.18 **For denomination, the most well-known way is that you decompose the denominating number into the numbers of which it is composed, and you take them as denominators. Then you divide what you want to denominate by them, resulting in the answer. One obtains its value by means of the ratio of these parts to the denominators serving as divisors.**

123.22 For example, suppose someone said, "Denominate eleven with fifteen". We decompose the fifteen, the denominating number, into the numbers from which it is composed, which are five and three. We put them below a line and we divide the eleven first by the three, and we put the remainder above it. And we divide the quotient by the five, which is smaller than it. We put it above it, so it yields this figure: $\frac{2\ 3}{3\ 5}$. Thus we form the ratio of the three to the five that is below it, and the two to the three that is below it, and we attach the ratio to the five. This gives the size of eleven with respect to fifteen, which is three fifths and two thirds of a fifth. So know it.

124.8 ***A lesser known way is that you divide the denominating number by the denominated number, and you denominate one with the result. Or you denominate one with the denominating number and you take this ratio of the denominated number. Or you multiply the denominated number by some [convenient] number and you divide the result by the denominating number, and [you divide] the result by that multiplied number.***[66]

124.12 An example of the first [way]: suppose someone said, "Denominate four with twelve". We divide the twelve by the four, and we denominate one with the result. This gives the required number, which is a third.

124.14 An example of the second: suppose someone said, "Denominate nine with fifteen". We denominate one with the fifteen, to get a third of a fifth. We take from the nine a third of its fifth. The result is the required number, which is three fifths.

124.17 And an example of the third: suppose someone said, "Denominate ten with sixteen". We multiply the ten by whatever number we wish, such as eight, to get eighty. We divide it by the sixteen, and the result of that by the eight, to get the required number, which is five-eighths. So know it.

124.20 **To decompose numbers requires that some preliminary remarks be learned. They are:**

125.1 **Every number that does not begin with units has a tenth and a fifth and a half, [the latter] being a characteristic of every even number.** For example, fifty and similar numbers. Its half is twenty-five, its fifth is ten, and its tenth is five.

[66] Copied from (Ibn al-Bannāʾ 1994, 267.11-14).

125.5 **If it begins with a five, it has a fifth. If it begins with units, and if it is even, then one casts out numbers using one of the three moduli,** IN THE SENSE OF WHAT IS SAID ABOVE ON CASTING OUT. **If it is cast out entirely by nines, then it has a ninth and a sixth and a third.** For example, thirty-six is cast out entirely by nines. Its ninth is four, its sixth is six, and its third is twelve.

125.11 **If the remainder is three or six, then it has a sixth and a third.** An example with a remainder of three is sixty-six, and similar numbers. Its sixth is eleven and its third is twenty-two. An example with a remainder of six is forty-two, and similar numbers. Its sixth is seven and its third is fourteen.

125.16 **If the remainder is something else, then cast out eights. If it is cast out entirely, then it has an eighth and a fourth.** For example, sixty-four is cast out entirely by eights. Its eighth is eight and its fourth is sixteen.

125.18 **If the remainder is four, then it has a fourth.** For example, sixty-eight. Its fourth is seventeen.

125.20 **And if the remainder is something else, then cast out sevens. If it is cast out entirely, then it has a seventh.** For example, fourteen is cast out entirely by sevens, and its seventh is two.

126.2 **If it is not cast out entirely, then it only has a half and its half is odd, [and you then] look for deaf parts.** For example, twenty-six is not cast out entirely by any of the three moduli, so it only has a half, and its half is thirteen, which is an odd number. So know it.

126.5 **If [the number] is odd, cast it out by two numbers, nine and seven. If it is cast out entirely by nines, then it has a ninth and a third.** For example, eighty-one is cast out entirely by nines. Its ninth is nine and its third is twenty-seven.

126.9 **If the remainder is three or six, then it has a third.** An example with a remainder of three is thirty-nine. Its third is thirteen. An example with a remainder of six is one hundred twenty-three. Its third is forty-one.

126.12 **And if the remainder is some other number, then cast out sevens. If it is cast out entirely, then it has a seventh.** For example, seventy-seven is cast out entirely by sevens, and its seventh is eleven.

126.14 **If it is not cast out entirely, then look for deaf parts by dividing by them; and you continue dividing the required number to be decomposed by deaf parts until you arrive at a number that divides into it, or you get a number whose square is greater than your given number, or [equivalently, that] the result from the division is equal to or less than the divisor and leaves a remainder after the division. At this point you know that it is one of the deaf parts, and the denomination is formed from it.**

127.1 For example, suppose we want to decompose two hundred twenty-one. We find that it is not cast out entirely by the two moduli, so we need to divide it by deaf parts, as mentioned. We find that it is not divisible by eleven, which is the first of them. But it is divisible by thirteen, and the result of the division is seventeen. So it is composed from thirteen by

[67] The text mistakenly has "a square of the dividend".

seventeen. If a square of the thirteen had been greater than the dividend,[67] we would have also known that it is deaf. If the result had been equal to the divisor thirteen, or if it had been less, with a remainder from the dividend, we would also have known that it is deaf. We denominate it with separate pieces, saying something like “three parts of this” or “a hundred twenty parts of that” or something similar.

127.9 **Subsection on finding deaf parts.**

127.10 **This process is called the sieve. Here you write the odd numbers beginning with three. Then, for each of these numbers, you count off its successors by the number of units in it. When this is done, the next number is composite, and it counts that [first] number. You continue to do this until you reach a number whose square is greater than the last number in the sieve, at which point you know that the work is finished. Each marked number is composite and each unmarked number is deaf.**

127.15 For example, we write down the odd numbers from three consecutively, as mentioned, in a table, as in this figure:

19	17	$\overline{15}$	13	11	$\overline{9}$	7	5	3
37	$\overline{35}$	$\overline{33}$	31	29	$\overline{27}$	$\overline{25}$	23	$\overline{21}$
$\overline{55}$	53	$\overline{51}$	$\overline{49}$	47	$\overline{45}$	43	41	$\overline{39}$
73	71	$\overline{69}$	67	$\overline{65}$	$\overline{63}$	61	59	$\overline{57}$
$\overline{91}$	89	$\overline{87}$	$\overline{85}$	83	$\overline{81}$	79	$\overline{77}$	$\overline{75}$
109	107	$\overline{105}$	103	101	$\overline{99}$	97	$\overline{95}$	$\overline{93}$
127	$\overline{125}$	$\overline{123}$	$\overline{121}$	$\overline{119}$	$\overline{117}$	$\overline{115}$	113	$\overline{111}$
$\overline{145}$	$\overline{143}$	$\overline{141}$	139	137	$\overline{135}$	$\overline{133}$	131	$\overline{129}$

128.1 If we want to know which among them is composite with three, we count it off from its cell, ending at the cell of the seven, so the nine that is next is composite with the three. So we put a mark above it. Similarly, we count beginning with the cell of the nine, ending with the cell of the thirteen. So the fifteen that is next is also composite with the three, and we put a mark above it. Do this until the end of the sieve. Likewise, do this with the five and the seven, but not with the nine, since it is composite, and not with any other composite number. This ends in our example with counting by thirteen. We know that the work is finished because its square is one hundred sixty-nine, which is greater than one hundred forty-five, which is the last number in the prescribed sieve.

128.9 And if we wish, we can work with a sieve greater or smaller than this one, since the method is the same for all of them. Every number in this sieve that is marked is composite, and every unmarked number is deaf, as we noted. These deaf parts are counted only by one, as we recall from the first part of the book.[68] We cannot find a number which, when multiplied by a number gives, for example, thirteen or one hundred fifty-one and the like.

[68] On prime numbers, at 66.7.

So if someone said to you, “Of which numbers is thirteen composed”, you would say thirteen by one. Multiplication by one is not duplication, as mentioned before. And this is the answer to similar [questions], so know it.

129.1 **Section Six, on restoration and reduction.**

129.2 **Restoration is reconstitution (*iṣlāḥ*), and reduction is its opposite. The purpose of restoration and reduction is to know what to multiply by a number to produce the required outcome. Restoration is only for taking the smaller to the greater, and reduction is for the opposite**, THAT IS, FROM THE GREATER TO THE SMALLER.

129.6 An example of restoration: suppose someone said, “By how much must one restore eight, for example, so that it gives nineteen?” An example of reduction: suppose someone said, “By how much must one reduce fifty, for example, so that it gives six?”

129.8 **To work out restoration, you divide the restored number by the number to be restored, which gives the required number.** For example, suppose someone said, “By how much must one restore three so that it gives six?” You divide the six, the restored number, by the three, the number to be restored, resulting in two. If you multiply that by the three it becomes six, which is the required number.

129.12 **To work out reduction, denominate the reduced number with the number to be reduced. This results in the answer.** For example, suppose someone said, “By how much must one reduce eight so that it yields three?” You denominate the three, the reduced number, with the eight, the number to be reduced. This gives three eighths. If you multiply the three eighths by the eight, it becomes three, which is the answer. This is the end of the first chapter, with God’s benediction and His good guidance.

131.1 **Chapter Two, on fractions.**

133.1 **The fraction is the ratio between two numbers that are a part or parts. The ratio between the part and its name is called a fraction.**

133.3 For example, three and six. The relation that arises with the ratio of the smaller to the greater is called the fraction. It cannot be named in terms of the three alone because of the separation, and likewise it cannot be named in terms of the six alone; nor in terms of [the two words] taken together, since the ratio is not a sensible object but is the idea of a specific intelligible object. It is called a fraction, like terrain with fractures that ascends and descends; or also like the surface, the solid, and the line, and there is nothing of this in discrete quantities except similar abstractions. Furthermore, the fraction consists of names, like “the half” and others that will be presented below, almighty God willing.

133.10 **We intend to present calculation with fractions in six sections.**

134.1 **Section One, on the names of fractions and numerating them.**

134.2 **Ten fractions have simple names. The first is the half, which is the greatest**, WHOSE FIGURE IS $\frac{1}{2}$. **Then the third**, WHOSE FIGURE IS $\frac{1}{3}$. **Then the fourth**, WHOSE FIGURE IS $\frac{1}{4}$. **Then the fifth**, WHOSE FIGURE IS $\frac{1}{5}$. **Then the sixth**, WHOSE FIGURE IS $\frac{1}{6}$. **Then the seventh**, WHOSE FIGURE IS $\frac{1}{7}$. **Then the eighth**, WHOSE FIGURE IS $\frac{1}{8}$. **Then the**

ninth, whose figure is $\frac{1}{9}$. **Then the tenth**, whose figure is $\frac{1}{10}$. **Then the "part"**, of which there are many types. You can say a part of eleven, whose figure is $\frac{1}{11}$, and a part of seventeen, whose figure is $\frac{1}{17}$, and so on.

134.8 **You can form the dual and plural of these fractions. Adding terminates for each fraction when you have [one] less than the name of the part.** The name is the greater of the two numbers in the ratio of one of them to the other. For example, we say a fourth and two fourths and three fourths, but we do not say four fourths. Likewise we say a seventh and two sevenths and three sevenths and four sevenths and five sevenths and six sevenths, but we do not say seven sevenths. So know it.

135.1 **You can attach these simple names to one another so that the resulting name is a combination of two or more names.** For example, we say two eighths and a seventh of an eighth, whose figure is $\frac{1\ 2}{7\ 8}$, and likewise for similar examples. Another example is eight ninths and four sevenths of a ninth and two sixths of a seventh of a ninth and a third of a sixth of a seventh of a ninth, whose figure is $\frac{1\ 2\ 4\ 8}{3\ 6\ 7\ 9}$, and likewise for similar examples.

135.8 **Numeration is that you reduce all of what is given to you in a particular problem to a finer fraction in it...**[69] *Know that the finer fraction in it is the part named with respect to all the denominators of the problem.*

135.10 *When the fraction is related, like five-sixths and four-fifths of a sixth and two-thirds of a fifth of a sixth,* whose figure is $\frac{2\ 4\ 5}{3\ 5\ 6}$, *we multiply what is above the first denominator by the denominator that follows it. That means to relinquish its name, which is sixths, in favor of fifths of sixths, since the second denominator is a number of what is in the one of the first denominator in units. So we add it with the four that is above it, since they are fifths of sixths, then we multiply that by the three, the third denominator, which is the number of what is in the one of the second denominator in units. This results in thirds of fifths of sixths. We add it with the two, since it is thirds of a fifth of a sixth. This gives the numerator of the problem, which is how many thirds of fifths of sixths there are in it,* and that is eighty-nine. *The multiplication of the denominators one by the other*, which is ninety, *is what is in the whole one of these parts, and a part of one of them is a finer fraction in the problem.*

136.8 *If the fraction is distinct, like five-sixths and four-fifths*, whose figure is $\frac{4}{5}\,\frac{5}{6}$, *we multiply the five-sixths by five, the denominator of the fifths, and that is the number of what is in the one sixth of fifths, so it becomes fifths of sixths. And we multiply the four-fifths by six,* the denominator of the sixths, *and that is the number of what is in the one fifth of sixths. So it becomes fifths of sixths. The sum of these two numerators are parts of thirty, which is sixths of fifths or fifths of sixths*, both being equal.

137.1 *The fraction may be portioned, in which the fractions are taken one of another, and which is how they are described in the expression. An example is three fourths of five sixths,* whose figure is $\frac{5\,\bullet\,3}{6\,\bullet\,4}$. *We multiply the three by the five, giving fifteen fourths of a sixth, or sixths of a fourth, which are parts of twenty-four parts of the unit. This is because five of the sixths, of which we want three of its fourths, [is found] by taking its fourth, which is five fourths of a sixth, and multiplying it by three, giving fifteen fourths of a sixth, or by taking a fourth of three times it. And three times it is its multiplication by three, giving*

[69] This sentence continues at 137.11.

fifteen sixths, and a fourth of that is fifteen fourths of a sixth, which is also fifteen sixths of a fourth. So three fourths of five sixths is five sixths of three fourths, since multiplying three by five is like multiplying five by three.[70] So know it.

137.11 **…and this is different for different [kinds] of fractions, that is, the numeration. There are five kinds [of fractions]: simple, related, distinct, portioned, and excluded.**

137.13 **The numerator of a simple fraction is what is above it.** For example, we said "a seventh": its numerator is the one that is above the line. Likewise for combined fractions, like if we said, "a third of a seventh": its numerator is also the one that is above the line.

138.1 **The numerator of a related fraction is what is above the first denominator multiplied by the next denominator, then adding to the end of the line; or, it is what is above the first denominator multiplied by the denominators after its denominator, and what is above the second denominator also multiplied by the denominators after its denominator, and so forth until the line is completed, then adding them together.**

138.5 For example, suppose someone said, "Numerate five eighths and four sevenths of an eighth and three fifths of a seventh of an eighth and two thirds of a fifth of a seventh of an eighth", whose figure is $\frac{2\ 3\ 4\ 5}{3\ 5\ 7\ 8}$. We multiply the five that is above the first denominator by the seven, the second denominator, and we add the four that is above it, giving thirty-nine. We also multiply that by the five, the third denominator, and we add the three that is above it, giving one hundred ninety-eight. We likewise multiply it by the three, the fourth denominator, and we add the two that is above it to it, giving five hundred ninety-six, which are eighths of sevenths of fifths of thirds. This is the numerator, and its figure is 596.

138.12 For the second way, we multiply the five that is above the first denominator by the other denominators except its denominator, as mentioned. This results in five hundred twenty-five. Keep it in mind. Then we multiply the four that is above the second denominator by the denominators after it, resulting in sixty. Keep this in mind as well. Then we multiply the three that is above the third denominator by the three, the fourth denominator, and we add what is above it, since there is nothing left after that denominator, giving eleven. We add it with the two remembered numbers, which gives five hundred ninety-six, which is the numerator, as before.

139.1 **The numerator of a distinct fraction is found by multiplying the numerator of each part by the other denominators, then adding them together.**

139.2 For example, suppose someone said, "Numerate five sevenths and half a seventh and four sixths". We put them below two lines as in this figure: $\frac{4}{6}\ \frac{1\ 5}{2\ 7}$. We find that the first part is related, so we take its numerator in the manner shown above, to get eleven. We multiply it by the six, the denominator of the second part, resulting in sixty-six. Keep it in mind. We find that the second part is simple, so its numerator is the four that is above the denominator. We likewise multiply it by the denominators of the first part, giving fifty-six. We add this with the remembered number to get a total of one hundred twenty-two, which is sevenths of halves of sixths. This is the numerator, and its figure is 122.

[70] Copied from (Ibn al-Bannāʾ 1994, 272.14-273.19).

139.10 **The numerator of a portioned fraction is found by multiplying what is above the line, each one by the next.**

139.11 For example, suppose someone said, “Numerate seven-ninths of five-sixths of three-tenths”. We put them on a line and we always separate them with marks, as in this figure: $\frac{3 \bullet 5 \bullet 7}{10 \bullet 6 \bullet 9}$.We multiply the seven that is above the first denominator by the five that is above the second denominator, and their product by the three that is above the third denominator, to get the numerator. That comes to one hundred five sevenths of sixths of tenths, and its figure is 105.

140.1 **The numerator of an excluded fraction of disconnected [type] is found the same way as for distinct fractions, by subtracting the smaller from the greater.**

140.2 For the disconnected [type], what follows the “less” is not taken from what precedes it, but rather is taken from one, and then it is removed. For example, if we said “a half less a third”, we mean less a third of one. So, after taking a third of the one, it is removed from the half. And when he said “the same way as for distinct fractions” he meant that you multiply what is before the “less” with what is after it in two different steps. We multiply the numerator of each part by the other denominator, and then we subtract the smaller from the greater.

140.8 For example, suppose someone said, “Numerate six eighths less a ninth of one”. We put them on a line as in this figure: $\frac{1}{9}$ $\frac{6}{8}$. We multiply the six, the numerator of what is before the “less”, by the nine, the denominator after it, giving fifty-four. From that we drop the product of the one, the numerator after the “less”, by the eight, the denominator before it, which is eight. The remainder is forty-six eighths of ninths, which is the numerator. Work out similar problems the same way, as explained above.

140.14 **For the connected [type], multiply the numerator of the diminished fraction by the numerator of the excluded fraction, and likewise multiply by their denominators, and then subtract the smaller from the greater.**

140.16 The connected [type] occurs when what follows the “less” is taken from what precedes it, without mediation. For example, suppose we said “a half less its third”. A third of the half is a sixth, so it is as if someone had said “a third”, or “a half less a sixth”, so it becomes disconnected.

141.1 Suppose someone said, “Numerate six-sevenths and half a seventh less its third”. We put them on a line, as in this figure: $\frac{1}{3}$ $\frac{1\ 6}{2\ 7}$. We take the numerator of what is before the “less”, which is the diminished fraction. We multiply it by the denominators of what is after the “less”, which is the excluded fraction, to get thirty-nine. Keep it in mind. Then we also multiply the numerator of the diminished fraction by the numerator of the excluded fraction, similarly giving the complement thirteen. We drop it from the remembered number. The remainder is twenty-six sevenths of halves of thirds, and it is the numerator.

141.7 Supplementary remark. ***If there are repeated deletions in which each of them has the conjunction “and”, it is with respect to the first [term]. And in relation to the first [term],***

[71] Copied from (Ibn al-Bannāʾ 1994, 275.8-9).

whether they are all connected to or detached from the diminished fraction, [treat them as] distinct fractions excluded from the diminished fraction.[71]

141.11 A first example: suppose someone said, "Numerate five and a third less its fourth and less its seventh and less its fifth". We put it down on a line like this figure: $\frac{1}{5} \searrow \frac{1}{7} \searrow \frac{1}{4} \searrow \frac{1}{3}\, 5$. The fourth and the seventh and the fifth are distinct fractions excluded from the five and a third, which is the diminished part, and they are all connected to it. Removing the second and third particles of exclusion, the figure in this problem becomes $\frac{1}{5}\,\frac{1}{7}\,\frac{1}{4} \searrow \frac{1}{3}\, 5$. We work it out as before, with the result that the numerator is nine hundred twelve thirds of fourths of sevenths of fifths, and its figure is 912.

142.6 A second example: suppose someone said in the previous problem, "less a fourth of one and less a seventh of one and less a fifth of one". Then these are distinct fractions removed from the diminished part, which are all detached. Write them also as if they were connected and work it out as before. The result is that the numerator is one thousand nine hundred ninety-one thirds of fourths of sevenths of fifths, and its figure is 1991.

142.10 If the exclusions are repeated without a coordinating conjunction,[72] in such a way that each [term] is excluded from the one before it, whether they are detached or connected, *then we take the [last] excluded fraction and the diminished fraction [before it], as in the other problem, and we work it out as shown above*, whether they are connected or detached. *The numerator we get is excluded from the one before it. Then we work that out similarly. The result is the numerator excluded from the one before it. Continue like this to the first [fraction].*[73] If we wish, we can work it out for disconnected fractions as in the two preceding cases.

143.1 *Whenever some of them are disconnected and others are connected, then the connected fractions must be transformed to their disconnected form when writing it down, so that all of them become disconnected. For instance, five sixths less three of its fourths is five sixths less three fourths of five sixths. Now what is written after the "less" is a disconnected fraction.*[74]

143.5 **And if there is a whole number before the fractions in a problem, it is multiplied by the denominators and then added with the numerator to become fractions.** For example, suppose someone said, "Numerate five and five sixths and three fourths of a sixth". So we put it down on a line like this: $\frac{3}{4}\,\frac{5}{6}\, 5$. We multiply the whole number five by the six, the first denominator, and what results by the four, the second denominator. This gives the result one hundred twenty. We add it with the numerator of the fraction, which is twenty-three. This gives a total of one hundred forty-three sixths of fourths, and its figure is 143, which is the numerator.

143.13 **And if it is after them, multiply the numerator by it,** SINCE THE FRACTIONS ARE PORTIONS OF IT. For example, suppose someone said, "Numerate four sevenths and six eighths of ten". We put it down on a line like this: $10\, \frac{6}{8}\, \frac{4}{7}$. The numerator of the fractions, as shown before, is seventy-four. We multiply it by the whole number ten to get seven hundred forty sevenths of eighths, which is the numerator, and its figure is 740.

[72] The coordinating conjunction is the word "and" (*wa*).
[73] Copied from (Ibn al-Bannāʾ 1994, 275.14-16).
[74] Copied from (Ibn al-Bannāʾ 1994, 276.3-5).

144.4 **And if it is in the middle, it is either attached to what is before it, so it is on the end, or it is attached to what is after it, so it precedes it. You numerate it according to one of the two cases, with the remaining [fractions] as distinct in the latter case, and in the former case one multiplies by the numerator of the remaining [fractions].**

144.7 "ATTACHED TO WHAT IS BEFORE IT" MEANS THAT ONLY THE FIRST FRACTION IS TAKEN OF THE WHOLE NUMBER, SO IT IS ONE PART. THE REMAINING FRACTION IS THE DISTINCT PART. YOU MULTIPLY THE NUMERATOR OF EACH PART BY THE DENOMINATOR OF THE OTHER AND YOU ADD THE RESULTS.

144.10 For example, suppose someone said, "Numerate four ninths of five, and three sixths". We put down the problem on a line like this: $\frac{3}{6}\,5\,\frac{4}{9}$. We multiply the four that is above the nine by the whole number five, giving twenty, which is the numerator of the first part. Then we multiply it by the six, the denominator of the second part, giving one hundred twenty. Keep it in mind. Then we multiply the three, the numerator of the second part, by the nine, the denominator of the first part, giving twenty-seven. We add it with the remembered number to get one hundred forty-seven sixths of ninths, which is the numerator. Its figure is 147.

145.1 AND THE "ATTACHED TO WHAT IS AFTER IT" [MEANS] THAT THE FIRST FRACTION IS TAKEN OF THE WHOLE NUMBER AND OF THE FRACTION AFTER IT. SO THE WHOLE NUMBER IS ATTACHED TO WHAT IS AFTER IT, AND IT PRECEDES IT. SO YOU NUMERATE IT WITH IT AND YOU MULTIPLY THAT BY THE NUMERATOR OF THE REMAINING [FRACTION], WHICH IS THE FIRST FRACTION, SINCE IT IS A PORTION OF IT.

145.4 For example, suppose someone said, "Numerate two thirds of seven and four sevenths". We put it down like this: $\frac{4}{7}\,7\,\frac{2}{3}$. We multiply the whole number seven by the seven, the denominator of the fraction, and we add the four that is above it and we multiply that by two, the numerator of the remaining [fraction], which is the first fraction. This gives one hundred six thirds of sevenths, which is the numerator, and its figure is 106.

145.8 *One understands from this that for every fraction, or more than one fraction, that is taken only of a whole number, the whole number comes after it. So we numerate it with it as one part. And the other fraction which is not taken of it we regard as distinct. And for every fraction, or more than one fraction, together with the whole number, if the whole number precedes it, then we numerate it with it as one part, and one multiplies its numerator by the numerator of the fraction taken of that whole number and the fraction that is with it.*

145.13 *Any remaining fractions in the problem other than those taken of the whole number and what is with it are distinct fractions. So the whole number and what is with it and what is taken of that is one part, and every fraction among the distinct fractions is a part. So one multiplies the numerator of each fraction by the denominator of the other and we add the results.*[75]

146.3 **Common divisors between the numerator and denominators should be removed.** We mentioned the way to do this by decomposition in the section on division.[76] *For portioned fractions in particular one should remove the common divisors before the numeration,*

[75] Copied from (Ibn al-Bannāʾ 1994, 276.16-277.2).
[76] At 123.3 above.

since the numbers ascribed to the numerator are above the line and the numbers ascribed to the denominators are below the line. So we drop the repeated [factors] from both of them.[77]

147.1 **Section Two, on adding and subtracting fractions.**

147.2 **To work out the addition, you multiply the numerator of each line by the denominators of the other, and you divide the sum by the denominators. And for subtraction you drop the smaller from the greater before dividing by the denominators.**

147.4 An example of addition: suppose someone said, “Add three and four fifths and six eighths to four tenths and three eighths of a tenth and half an eighth of a tenth”. We put the addend on a line and the augend on another line below it, like in this figure:

$$\frac{6}{8}\,\frac{4}{5}\,3$$

$$\frac{1\;\;3\;\;\;4}{2\;\;8\;\;10}$$

We numerate the upper fraction as before. Its numerator is one hundred eighty-two. We multiply it by the denominators of the lower [fraction], to get twenty-nine thousand one hundred twenty. Keep it in mind. Then we multiply the numerator of the lower [fraction], which is seventy-one, by the denominators of the one above, resulting in two thousand eight hundred forty. We add it with the remembered number to get thirty-one thousand nine hundred sixty, and its figure is 31960. We divide it by the denominators of the two lines. The result is the required number, and that is four and nine tenths and seven eighths of a tenth and half an eighth of a tenth, and its figure is $\frac{1\;\;7\;\;\;9}{2\;\;8\;\;10}4$. And its answer is given by five.

147.15 Another example, of subtraction: suppose someone said, “Subtract seven tenths of two less a third of one from four and three fourths of five sixths”. We put the minuend down on a line and the subtrahend on another line below it, like with the addend, as in this figure:

$$\frac{5\;\bullet\;3}{6\;\bullet\;4}\,4$$

$$\frac{1}{3}\quad 2\,\frac{7}{10}$$

148.3 We numerate the upper [fraction] as before. Its numerator is one hundred eleven. We multiply it by the denominators of the lower [fraction], resulting in three thousand three hundred thirty. Keep it in mind. Then we multiply the numerator of the lower [fraction], which is thirty-two, by the denominators of the upper [fraction], resulting in seven hundred sixty-eight. We drop it from the remembered number, leaving the required number, which is two thousand five hundred sixty-two, and its figure is 2562. We divide it by the denominators

[77] Copied from (Ibn al-Bannāʾ 1994, 277.4-6).

of the two lines. The result is the answer, which is three and five tenths and three sixths of a tenth and half a sixth of a tenth, and its figure is $\frac{1\ 3\ \ 5}{2\ 6\ 10}3$. And its answer is given by subtraction.

149.1 **Section Three, on multiplying fractions.**

149.2 **This is the portioning of one of two fractions by the amount of the other.** This contrasts with whole numbers or the duplication of fractions by the amount of a whole number. If the multiplier is a whole number and the multiplicand is a fraction, or vice versa, then either one extracts the whole number by the amount of the fraction, or one duplicates the fraction by the amount of the whole number.

149.6 **To work it out,** that is the multiplication of fractions**, you multiply the numerator of each line by the numerator of the other, and you divide the result by the denominators.**

149.8 For example, suppose someone said, “Multiply three fourths and a third by three ninths and four sixths of a ninth and a fifth of a sixth of a ninth”. We put the multiplier on a line and the multiplicand below it on another line, as in this figure:

$$\frac{1}{3}\,\frac{3}{4}$$

$$\frac{1\ \ 4\ \ 3}{5\ \ 6\ \ 9}$$

We multiply the numerator of the upper [fraction], which is thirteen, by the numerator of the lower [fraction], which is one hundred eleven. It results in one thousand four hundred forty-three, and its figure is 1443. We divide it by the denominators, resulting in the required number, which is four ninths and a fourth of a fifth of a sixth of a ninth, and its figure is $\frac{1\ 0\ 0\ 4}{4\ 5\ 6\ 9}$. And its answer is given by one.

150.2 Another example: suppose someone said, “Multiply a third of four and an eighth by a fifth of two thirds of ten”. We put the problem down on two lines as before, like in this figure:

$$\frac{1}{8}\,4\,\frac{1}{3}$$

$$10\,\frac{2\ \bullet\ 1}{3\ \bullet\ 5}$$

We multiply the numerator of the upper [fraction], which is thirty-three, by the numerator of the lower [fraction], which is twenty, resulting in six hundred sixty, and its figure is 660. We divide it by the denominators, resulting in the required number, which is one and five sixths, and its figure is $\frac{5}{6}1$. And its answer is given by two.

151.1 **Section Four, on division and denomination.**

151.2 **To work these out you multiply the numerator of each line by the denominators of the other, and you divide the result for the dividend by the result for the divisor, or you denominate.**

151.4 An example of division: suppose someone said, “Divide six and a third by four fifths of seven eighths of three”. We put the dividend on a line and the divisor on another line below it, as in this figure:

$$\frac{1}{3}\,6$$

$$3\,\frac{7 \bullet 4}{8 \bullet 5}$$

We multiply the numerator of the dividend, which is nineteen, by the denominators of the divisor, resulting in seven hundred sixty. Its figure is 760, which is the result for the dividend. Keep it in mind. Then we multiply the numerator of the divisor, which is eighty-four, by the denominators of the dividend, resulting in two hundred fifty-two. Its figure is 252, which is the result for the divisor. We divide the remembered number by it. The result is the required number, which is three and a seventh of a ninth, and its figure is $\frac{1\ 0}{7\ 9}\,3$. And its answer is given by four.

151.14 Another example, of denomination: suppose someone said, “Denominate three and a fourth less two ninths of it with six and two eighths and three fifths”. We put down the problem as in this figure:

$\frac{2}{9}$ ﻻ $\frac{1}{4}$ 3

$$\frac{3}{5}\,\frac{2}{8}\,6$$

We multiply the numerator of the denominated number, which is ninety-one, by the denominators of the denominating number to get the result of three thousand six hundred forty, and its figure is 3640. Keep it in mind. Then we multiply the numerator of the denominating number, which is two hundred seventy-four, by the denominators of the denominated number to get the result of nine thousand eight hundred sixty-four, and its figure is 9864. We denominate the remembered number with it to get the required number, which is fifty parts of one hundred thirty-seven parts and five ninths of a part of one hundred thirty-seven parts, and its figure is $\frac{5\ \ 50}{9\ 137}$. And its answer is given by subtraction.

152.9 **When the denominators of the two lines are equal, you divide the numerator by the numerator or you denominate with it, without multiplying by the denominators.**

152.11 An example of division: suppose someone said, “Divide eight and nine tenths and two thirds of a tenth by five tenths and a third of a tenth”. We put down the problem like this:

$$\frac{2\ \ 9}{3\ \ 10}\,8$$

$$\frac{1\ \ 5}{3\ \ 10}$$

We divide the numerator of the dividend, which is two hundred sixty-nine, by the numerator of the divisor, which is sixteen. The result is the required number, which is sixteen and six eighths and half an eighth, as in this figure: $\frac{1\ 6}{2\ 8}\,16$.

153.3 Another example, with denomination: suppose someone said, “Denominate two and a third with six and two thirds”. We put down the problem like this:

$$\frac{1}{3}\,2$$

$$\frac{2}{3}\,6$$

We denominate the numerator of the denominated number, which is seven, with the numerator of the denominating number, which is twenty. This results in the required number, which is three tenths and half a tenth, and its figure is $\frac{1\ \ 3}{2\ \ 10}$.

153.7 **When the two numerators are equal, you divide the denominators of the divisor by the denominators of the dividend, or you denominate, without multiplying by the numerator**, since, if we multiply by the denominators, the dividend will be composed from its numerator and the denominators of the divisor, and the divisor will be composed from its numerator and the denominators of the dividend. The two numerators vanish when the common divisors are removed. This is also the cause for the previous method.

153.12 For example, *suppose someone said, “Divide five by five sixths”. We divide the six,* the denominator of the divisor, *by one.*[78] It gives six, since whenever the dividend or the divisor is a whole number it is its numerator, and its denominator is one.

153.15 Similarly, suppose someone said, “Denominate five sixths with five”. You denominate one with six, giving a sixth. So know it.

154.1 **Section Five, on restoration and reduction.**

154.2 **To work these out you divide the restored number by the number to be restored, or you denominate the reduced number with the number to be reduced, to get the required number.** When he said “required number” he meant what is multiplied by the number to be restored in order to restore, or by the number to be reduced in order to reduce.

[78] Copied from (Ibn al-Bannāʾ 1994, 279.15).

154.6 There are six problems of restoration. One of them is restoration of a fraction to a fraction. For example, suppose someone said, “By how much must we restore a half so that it gives nine tenths?” We divide the nine tenths, the restored number, by the half, the number to be restored. The result is the required number, which is one and eight tenths.

154.10 The second is restoration of a fraction to a whole number and a fraction. For example, suppose someone said, “By how much must we restore two sevenths and half a seventh so that it gives five and a half?” We work this out as shown above to get the required number, which is fifteen and four tenths.

154.13 The third is restoration of a fraction to a whole number. For example, suppose someone said, “By how much must we restore two thirds of five sevenths so that it gives ten?” We work it out as shown above to get the required number, which is twenty-one.

154.16 The fourth is restoration of a whole number to a whole number and a fraction. For example, suppose someone said, “By how much must we restore five so that it gives ten and four sixths?” We work it out as shown above to get the required number, which is two and a tenth and a third of a tenth.

155.2 The fifth is restoration of a whole number and a fraction to a whole number. For example, suppose someone said, “By how much must we restore four and three tenths and half a tenth so that it gives eight?” We work it out as shown above to get the required number, which is one and seventy-three parts of eighty-seven parts.

155.6 The sixth is restoration of a whole number and a fraction to a whole number and a fraction. For example, suppose someone said, “By how much must we restore three and three fifths less a third of one so that it gives twelve and three fifths?” We work it out as shown above to get the required number, which is three and six sevenths. So understand it and you will succeed, almighty God willing.

155.11 And for reduction there are also six problems. One of them is reduction of a fraction to a fraction. For example, suppose someone said, “By how much must we reduce seven tenths so that it becomes a third?” We work it out as described. You denominate the third, the reduced number, with the seven tenths, the number to be reduced. This results in the required number, which is three sevenths and a third of a seventh.

155.16 The second is reduction of a whole number to a whole number and a fraction. For example, suppose someone said, “By how much must we reduce eight so that it becomes two and a half?” We work it out as shown above to get the required number, which is two eighths and half an eighth.

155.19 The third is reduction of a whole number to a fraction. For example, suppose someone said, “By how much must we reduce ten so that it becomes three fourths?” We work it out as shown above to get the required number, which is three fourths of a tenth.

156.3 The fourth is reduction of a whole number and a fraction to a whole number and a fraction. For example, suppose someone said, “By how much must we reduce seven and a fourth so that it gives three and four sixths?” We work it out as shown above to get the required number, which is fourteen parts of twenty-nine parts and four sixths of a part of twenty-nine parts.

156.7 THE FIFTH IS REDUCTION OF A WHOLE NUMBER AND A FRACTION TO A WHOLE NUMBER. For example, suppose someone said, “By how much must we reduce eleven and nine tenths and four sevenths of a tenth so that it becomes five?” We work it out as shown above to get the required number, which is thirty-eight parts of ninety-three parts and eight ninths of a part of ninety-three parts.

156.12 THE SIXTH IS REDUCTION OF A WHOLE NUMBER AND A FRACTION TO A FRACTION. For example, suppose someone said, “By how much must we reduce two and a third so that it becomes a ninth?” We work it out as shown above to get the required number, which is three sevenths of a ninth. So know it and manage it.

157.1 **Section Six, on converting.**[79]

157.2 ***This section covers two kinds. The intent of one kind concerns only the name,*** *such as when it is said “five sixths and three fourths: how many tenths is it?” We want to denominate these two fractions by naming the fraction in tenths. So we work it out as described, resulting in one and five tenths and five sixths of a tenth, which is what we get from gathering these two fractions together after turning them into tenths. So we shifted the problem from naming in terms of sixths and fourths to naming in terms of tenths and its fractions. This is shifting one kind of fraction to another kind, and it is this kind that is intended in the book.*

157.9 ***The intent of the second kind is: how many of that name, taken as units, are in the whole [fraction]? Work out this kind the same way as for whole numbers. Whenever we want to convert it, we look back to see how it was described in the second type of multiplication.***[80]

157.12 *Suppose someone said, “Five sixths and three fourths: how many tenths are in it?” We multiply them by the whole number ten, resulting in one and five tenths and five sixths of a tenth, which is the answer. This is the amount that comes about from the tenth, which is fifteen tenths and five sixths of a tenth.*[81]

158.1 *Unlike for the first kind, this kind [of conversion] does not require division by the denominator of the converted fraction, since it is similar to the case in which someone said, “Five dirhams, how many tenths are in it?” We multiply the five by the ten to get the result of fifty, which is the answer.* ***The same [rule] can be applied to fractions.*** *Since this kind belongs to the section on multiplication, it is not mentioned* by the author *in the book. He mentioned only the kind specific to the section [on fractions].*[82]

158.7 **To work this out,** THAT IS, THE INTENDED KIND**, you multiply the numerator of the fraction to be converted by the denominator of the converted fraction. One divides the result first by the denominators of the fraction to be converted and then the result**

[79] Everything from 157.2 to 158.6 is slightly condensed from (Ibn al-Bannāʾ 1994, 279.18-280.20).
[80] At 95.6 above.
[81] Copied from (Ibn al-Bannāʾ 1994, 279.18-280.12).
[82] Copied from (Ibn al-Bannāʾ 1994, 280.16-20). Al-Hawārī added “by the author” and changed Ibn al-Bannāʾ’s “I mentioned” to “he mentioned”.

by the denominator of the converted fraction. The advantage of this kind is that one can convert a fraction into a finer fraction.

158.11 For example, suppose someone said, "Six eighths and four tenths, how many ninths are in it?" We put the fraction to be converted on a line, and the denominator of the converted fraction on a line below it, as in this figure:

$$\frac{4}{10}\ \frac{6}{8}$$

$$\frac{\cdot}{9}$$

We multiply the numerator of the fraction to be converted, which is ninety-two, by the nine, the denominator of the converted fraction, resulting in eight hundred twenty-eight, and its figure is 828. We divide it first by the denominators of the fraction to be converted, then what is left by the denominator of the converted fraction, resulting in the required number. This is one and a ninth and three tenths of a ninth and four eighths of a tenth of a ninth, and its figure is $\frac{4\ \ 3\ \ 1}{8\ 10\ 9}\,1$. The same works for similar [examples].

159.5 This completes the chapter, with God's blessing.

161.1 **Chapter Three, on roots. Related to this, we cover what we intend on this topic in four sections.**

163.1 **Section One, on taking a root of a whole number and a root of a fraction.**

163.2 **These are divided into two varieties, rational and surd.** ***A rational [root] is any number whose ratio to one is known. This can be a whole number, a fraction, or a whole number and a fraction.***

163.4 ***A surd [root] is one whose ratio to one is unknown.*** *For example, a root of ten, a root of a half, and a root of ten and a half. Surds come in two varieties: those that are expressed with [the word] "root" once, like those just mentioned, and which are called rational in square, and those expressed by "root" more than once, like a root of a root of ten, and which are called medial.*[83]

163.9 **And the root is any number, which multiplied by itself, results in the number whose root is sought.** Examples have been given above.

163.11 In language, it is the origin of everything. He said "root" (*jadhr*) with a "dh" and either an "a" or an "i" as the vowel for the "j". Our professor al-shaykh Abū l-ʿAbbās [Ibn al-Bannāʾ], God be satisfied with him, told me that the "a" is more appropriate.

163.14 *The ranks in a whole number "have a root" and "do not have a root" for each successive place. This is evident by examination for the units and tens. The hundreds have a root because they come from multiplying the tens by themselves, and the thousands do not have a root because they are in relation to the hundreds in the position that the tens are*

[83] Copied from (Ibn al-Bannāʾ 1994, 283.3-7).

in relation to the units, and similarly for what comes after that. A place is said to have a root if there is a number in it that has a root.[84]

164.3 *Some conditions for a number may indicate that it does not have a root. But if they do not hold it only implies that it may have a root. These are:*

164.5 *Any number that begins with a two or a three or a seven or an eight does not have a root.*

164.6 *Any number that begins with a one, and half of its tens is different from the number of hundreds in the even and the odd,*[85] *does not have a root*, like three hundred forty-one and four hundred sixty-one and the like.

164.10 *Any number that begins with a five, and its tens is not twenty, does not have a root*, like seventy-five and one hundred eighty-five and the like.

164.12 *Any number that begins with a six, and its tens are even, does not have a root*, like forty-six and three hundred twenty-six and the like.

164.14 *Any number that does not begin with six, and its tens are odd, does not have a root.*[86]

164.15 Indications for non-square numbers, including those given above, mean the same as knowing that the number does not have a root.

164.17 Indications for [the first digit of] a square are five numbers: a one, with the condition that its tens [place] has a half; a five, with the condition that its tens are twenty; and a six, with the condition that its tens are odd. This leaves nine and four. If one of these is the first digit, as mentioned, and its tens are odd, then it does not have a root, and if it is even then it might.

165.3 *Any number beginning with an odd number of zeros does not have a root*, like ten, twenty-one thousand, three thousand, and the like.

165.5 *Any number beginning with an even number of zeros, and for which the number [remaining after the zeros are deleted] does not have a root, [also] does not have a root*, like five hundred, thirty thousand, and the like.

165.8 *Any number that is not exhausted after casting out nines, and the remainder is not a one or a four or a seven, does not have a root,*[87] like four hundred twenty-five and the like.

165.11 Any number that is not exhausted after casting out eights, and the remainder is not a one or a four, does not have a root, like two hundred sixty-six and the like.

165.14 Any number that is not exhausted after casting out sevens, and the remainder is not a one or a two or a four, does not have a root, like three hundred forty-nine and the like.

166.1 **The way to take a root of a whole number is to count off the ranks with "root", "no root" to the end of the line. Once you arrive at the last "root", put a number**

[84] Copied from (Ibn al-Bannāʾ 1994, 283.8-11).
[85] I.e., the parity of half of the tens digit is different from that of the hundreds digit.
[86] Copied from (Ibn al-Bannāʾ 1994, 284.14-285.1).
[87] Copied from (Ibn al-Bannāʾ 1994, 285.2-285.6).

below it so that if you multiply it by itself, it cancels what is above it or it leaves the smallest possible whole number as remainder. Then you back up, double it, and put it below the place [marked] "no root". Now you look for a number to put below the "root" preceding it, so that when you multiply it by the doubled number one step back and then by itself, it either cancels what is above it or it leaves the smallest possible remainder. Then if it did not cancel, you do it again. You continue doubling the backed-up number and shifting until you have covered all the line. What you then have in the second line, before doubling, is the root.

166.9 **If something remains, denominate it with double the whole part of the root if it is equal to or smaller than the root; and if it is greater than the root, always add one to it, and two to double the root, then you denominate it with it and you add the denomination to the whole part. This gives the root that you multiply by itself, and it is an approximation of a root of the given number.**

166.13 For example, suppose someone said, "How much is a root of six hundred twenty-five?" We put it on a line like this: 625. The first place has a root, the second does not have a root, and the third has a root, as mentioned above. We then look for a number to put below the six, which is in a rank that has a root, such that when we multiply it by itself it either cancels the six or it leaves the smallest possible whole number as remainder. We find that it is two. We multiply it by itself, giving four. We drop it from the six, leaving two, which we put in its place. We back up the doubled two and put it below the place [marked] "no root", which is below the two. Then we look for a number to put below the [rank that] has a root that is before the backed up number below it, and that is below the five. We find it is five. It cannot be anything else.

166.21 If it were a six, you could only put a four or a six below it, and if it were a one, you could only put a one or a nine below it, and if it were a four, you could only put a two or an eight below it, and if it were a nine, you could only put a three or a seven below it. So know it.

167.3 So we multiply the five by the doubled four, giving twenty. The twenty-two above its head cancels the twenty, leaving two in its place. Then we likewise multiply it by itself, giving twenty-five, and above its head is twenty-five, which cancels it. The meaning of the five and the doubled two after it is twenty-five, which is the required number. Work it out the same way if there are more digits.

167.8 Another example: suppose someone said, "How much is a root of twenty?" We know that it does not have a rational root since it begins with one zero. We work it out as explained above, and we divide the remainder according to the calculation described above. We look for a number such that when we multiply it by itself, it cancels it or it leaves the smallest possible remainder. We find it is four. We multiply it by itself, giving sixteen. The remainder is four, which is equal to the root. We denominate it with its double, which gives a half. We add it to the whole part of the root, giving four and a half, which is a root of twenty by approximation.

167.14 Another example: suppose someone said, "How much is a root of fifty-four?" We work it out as before, to get seven for the whole part of the root and five for the remainder, which is smaller than the root. We denominate it with its double, giving two sevenths and half a seventh. We add it to the root, to get the sum of seven and two sevenths and half a seventh, which is the required root by approximation.

168.1 Another example: suppose someone said, “How much is a root of ninety-two?” We work it out as before to get nine for the whole part of the root and eleven for the remainder, which is greater than the root. So we add one to it, and two to double the nine, which is the root. We denominate the smaller with the greater and we add it to the root. The result is the required root, which is nine and three fifths. If we multiply the nine and three fifths, which is the approximated root, by itself, it results in ninety-two and four fifths of a fifth. The approximation is found with the added fraction.

168.8 **And if you want to refine the approximation, denominate it with double the root. Drop the result from the root, leaving a root whose square is closer to the number whose root is required than the first square.**

168.10 He said “denominate it”, by which he meant the fraction added to the square that was found by approximation; because taking the root by approximation can be obtained from a previous close smaller square as seen above, and it can also be obtained from a previous close greater square, which is what he meant when he said “denominate it, etc.”. The description of how to work it out is that we drop the number whose root is required from the square, and we denominate the remainder with double a root of the square, and we also subtract the result from a root of the square. The remainder is a root of the number by approximation.

168.16 If we want to take a root of ninety-two using the previous greater square, we make the greater square ninety-two and four-fifths of a fifth. If we drop the number from it, the remainder will be the fraction found by approximation. So we denominate it with double the root, as mentioned, to get half a sixth of a tenth. We drop it from the root, leaving nine and five-tenths and five-sixths of a tenth and half a sixth of a tenth. A square of this remainder is closer than the first square. So know it.

169.1 **There is another method of approximation, which is that you multiply the number whose root is required by a greater square number, and you take a root of the result by approximation, and you divide by a root of the multiplied square. The result is the approximated root.**

169.4 For example, suppose someone said, “How much is a root of twelve?” We multiply it by sixteen, for instance, resulting in one hundred ninety-two. We take its root to get thirteen and six-sevenths. We divide it by a root of sixteen. The result is a root of twelve by approximation, and that is three and three-sevenths and a fourth of a seventh.

169.8 *The condition “the smallest possible whole number”*[88] *[is needed] because if one diverges from the well-known method by working with fractions, then the remainder will be smaller than the remainder with whole numbers.*[89]

169.10 *And suppose someone said, “How much is a root of six hundred twenty-five?”* This example is from the earlier problem. *We put two and a half below the six, so its square is six and a fourth. This exceeds [the six]. So you take away the six with the six, and take away the fourth with the twenty-five, which is a fourth of a hundred. So all the numbers vanish, and the remainder begins with the last two zeros. We take one of them. Thus the two and a*

[88] From 166.3 and 166.16 above.
[89] Copied from (Ibn al-Bannāʾ 1994, 283.11-12).

half are tens, which is twenty-five. We double the two and a half to get five, and we place it below the tens. And we look for something to multiply by the doubled number. We find it to be nothing, since there are zeros above it. We put down a zero and we halve what we doubled, so it yields half of fifty.[90]

169.17 Another *example: suppose someone said, "How much is a root of seven hundred twenty-nine?" If we work it out with whole numbers, then the remainder from the seven, which is seven hundred, in its rank is three. And if we work it out with fractions, then the remainder is smaller. So if we put two and a half below it, three fourths of one remains in that rank, and three fourths of a hundred is seventy-five. We add it to the twenty-nine that is with it, to get one hundred four. Then we back up the doubled two and a half, which is five, and we look for a number to multiply by the five and by itself. We find it is two, and nothing remains of the number. So we halve what we doubled, which is fifty. Its half is twenty-five, so the total root is twenty-seven. Or we double the two, so it becomes fifty-four, and we take half of it.*[91]

170.6 Another example: suppose someone said, "How much is a root of a hundred?" The basic rule for this and similar problems is that we always take half of the number of zeros, and we add them to a root of the remaining number to get the root. The hundred begins with two zeros. We take one of them and we add it to a root of the remaining one, giving ten, which is a root of the hundred. So know it.

170.10 **To take a root of a fraction, you multiply the numerator by the denominator and you divide a root of the result by the denominator. If the numerator has a rational root and the denominator does too, then divide a root of the numerator by a root of the denominator.**

170.14 WITH REGARD TO TAKING A ROOT, I MEAN OF A FRACTION, THERE ARE FOUR TYPES. IN ONE OF THEM THE NUMERATOR HAS A RATIONAL ROOT AND THE DENOMINATOR DOES TOO. WORK IT OUT AS DESCRIBED.

170.16 For example, suppose someone said, "How much is a root of four-sixths and a sixth of a sixth?" Its figure is $\frac{1\ 4}{6\ 6}$. We take a root of the numerator, giving five. We divide it by a root of the denominator, which is six. The result is the required root, and that is five-sixths.

171.1 Another example: suppose someone said, "How much is a root of twelve and a fourth?" We take a root of the denominator, giving two. We divide seven, a root of the numerator, by it. The result is the required number, which is three and a half. And if we wish, we can work it out by the first method, since it is general, while this one is particular. So know it.

171.5 IN THE SECOND [TYPE] NEITHER OF THEM HAS A RATIONAL ROOT, SO WORK IT OUT BY THE FIRST METHOD.

171.6 For example, suppose someone said, "How much is a root of four-ninths and three sixths of a ninth?" Its figure is $\frac{3\ 4}{6\ 9}$. We multiply the numerator by the denominator to get one thousand four hundred fifty-eight. We take its root, which is thirty-eight and three parts of nineteen parts and half a part of nineteen parts. We divide it by the denominator. The result is the required number, which is thirteen parts of nineteen parts and three ninths of a

[90] Copied from (Ibn al-Bannāʾ 1994, 284.4-9).
[91] Copied from (Ibn al-Bannāʾ 1994, 283.13-284.3).

part of nineteen parts and five-sixths of a ninth of a part of nineteen parts and half a sixth of a ninth of a part of nineteen parts, and its figure is $\frac{1\ 5\ 3\ 13}{2\ 6\ 9\ 19}$.

171.13 In the third [type] the denominator has a rational root but the numerator does not have a rational root. For this type, if we wish, we can work it out by the first [method] or by the second.

171.15 For example, suppose someone said, "How much is a root of ten and seven-eighths and half an eighth?" Its figure is $\frac{1\ 7}{2\ 8}10$. If we want, we can work it out by the first method. We multiply the numerator by the denominator, resulting in two thousand eight hundred. We take its root, which is fifty-two and forty-eight parts of fifty-three parts and half a part of fifty-three parts. We divide it by the denominator to get the required number, which is three and sixteen parts of fifty-three parts and two eighths of a part and a fourth of an eighth of a part of fifty-three parts, and its figure is $\frac{1\ 2\ 16}{4\ 8\ 53}3$.

172.6 By the second method, we take a root of the numerator, which is thirteen and three parts of thirteen parts. We divide it by a root of the denominator. The result is the required number, which is three and four parts of thirteen parts, and its figure is $\frac{4}{13}3$. This method is closer than the first [method].

172.10 In the fourth [type] the numerator has a rational root and the denominator does not have a rational root, so work it out by the first method.

172.12 For example, suppose someone said, "How much is a root of four sevenths and half a seventh?" Its figure is $\frac{1\ 4}{2\ 7}$. We multiply the numerator by the denominator and we take a root of the result. This is eleven and two parts of eleven parts and half a part of eleven parts. We divide it by the denominator. The result is the required number, which is eight parts of eleven parts and five sevenths of a part and three fourths of a seventh of a part of eleven parts, and its figure is $\frac{3\ 5\ 8}{4\ 7\ 11}$.

173.2 Of the four types, a root of the first one is found exactly, and the other three are found by approximation. If we wish to approximate [more accurately] the root like we did for whole numbers, then follow the same procedure.

173.4 **To take roots of binomials and apotomes, you drop a fourth of a square of the smaller of the two terms from a fourth of a square of the greater one; you take a root of the remainder, you add it to half of the greater term, and you also subtract it from half of the greater term; and you drop a root on each of them. If the number whose root is required is a binomial, then its root is the sum of these two roots, and if it is an apotome, then its root is the difference between these two roots.**

173.10 I say we need to begin with an introduction to clarify binomials and apotomes and how to find them. After that we will take their roots, almighty God willing.

173.12 We say that *there are six binomials and six apotomes.*

173.13 *A binomial is a number and a root of a number, or a root of a number and a root of a number in which the two can only be joined with the coordinating conjunction.*[92] *For example, five and a root of three, and a root of five and a root of three.*

[92] The coordinating conjunction is the word "and" (*wa*).

173.16 *An apotome consists of two terms in which the smaller term is removed from the greater by the particle of exclusion.*[93] *For example, five less a root of three, and a root of five less a root of three.*

174.1 *Roots of the first three binomials and apotomes are closer to being rational in rank than roots of the other three. The first three can be distinguished from the others by multiplying the difference between the squares of the terms by a square of the greater one. If the result is a square, then it is one of the first three, and if it is not a square, then it is one of the second three.*

174.5 *The greater term is rational in the first and the fourth.* An example of the first is five and a root of twenty-one, and an example of the fourth is two and a root of two.

174.8 *The smaller is rational in the second and the fifth.* An example of the second is five and a root of forty-five, and an example of the fifth is five and a root of seventy-two.

174.11 *Neither of them is rational in the third and the sixth.* An example of the third is a root of ten and a root of eighteen, and an example of the sixth is a root of seven and a root of eight.

174.14 *It is necessary to recall their characteristics if we want to find them. If we subtract a square from a square and the remainder is not a square, and we join a root of the remainder with a root of the greater square, this gives the first binomial.* Or we make whatever number we wish the greater term, and [we make] the smaller a root of a surface whose two sides are in numerical ratio,[94] with the condition that it is not rational. However, this procedure is rare.

175.1 *We subtract a non-square number from a square such that the remainder is not a square, and we join a root of the remainder with a root of the square to get the fourth binomial.*

175.3 *We multiply two squares by their difference, which gives non-squares, and we join a root of the greater of the two results with a root of their difference to get the second binomial.*

175.5 *We multiply two squares by something other than their difference, which gives non-squares, and we join a root of the greater of the two results with a root of their difference to get the third binomial.*

175.7 *We add a square to a square so that the sum is not a square, and we join a root of the sum with a root of one of the two squares to get the fifth binomial.*

175.9 *We add a non-square number to a square so that the sum is not a square, and we join a root of the sum with a root of the added number to get the sixth binomial.*[95]

175.11 These are the six binomials. Now let us return to examples of their roots. Example: suppose someone said, “Eight and a root of sixty: how much is a root of that?” We work this out as described. We drop a fourth of a square of a root of the sixty, since it is the smaller, and that is fifteen, from a fourth of a square of the eight, since it is the greater, and that is sixteen. The remainder is one. We take its root, giving one. We add it to half of the eight,

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[93] The particle of exclusion is most often the word “less” (*illā*).

[94] I.e., they are commensurable.

[95] Copied from (Ibn al-Bannāʾ 1994, 287.19-288.19).

since it is the greater, giving five, and we drop it likewise from its half. The remainder is three. We drop a root on the five and the three. This gives a root of five and a root of three, which is the required number. The same rule applies to other [examples].

175.19 And suppose someone said, “Eight less a root of sixty: how much is its root?” We work this out as before. We drop a root of the three from a root of the five. A root of their difference is the required root, and that is a root of the five less a root of the three.

176.1 Another way to do it is that we drop a square of the smaller term from a square of the greater one, and we take a root of the remainder. We add it to the greater term and we take a root of half of the sum. And likewise, we subtract it from the greater term and we take a root of half of the sum. If the number whose root is required is a binomial, then its root is the sum of these two roots. If it is an apotome, then its root is the difference between the two roots.

176.6 Suppose someone said, “Eight and a root of fifty-five: how much is its root?” This is the first binomial. We take its root as before, to get a root of five and a half and a root of two and a half. It is called “one of the binomials”. Its apotome is the first apotome, and an apotome of its root is a root of its apotome, and it is called “one of the six apotomes”.

176.10 Suppose someone said, “Seven and a root of one hundred twelve: how much is its root?” This is the second binomial. We take its root as before to get, after the addition and subtraction, a root of a root of eighty-five and three fourths and a root of a root of one and three fourths. This is called “the first bimedial”, and its apotome is the second apotome. An apotome of its root is a root of its apotome, and it is called “a first apotome of a medial”.

176.15 Suppose someone said, “A root of thirty-two and a root of fourteen: how much is its root?” This is the third binomial. We take its root as before. This gives, after the addition and subtraction, a root of a root of twenty-four and a half and a root of a root of a half. This is called “the binomial of the second bimedial”, and its apotome is the third apotome. An apotome of its root is a root of its apotome, and it is called “a second apotome of a medial”.

176.20 Suppose someone said, “Seven and a root of thirty: how much is its root?” This is the fourth binomial. We take its root as before, to get three and a half and a root of four and three fourths, taking its root, and three and a half less a root of four and three fourths, taking its root. It is called “the major”, and its apotome is the fourth apotome. And an apotome of its root is a root of its apotome, and it is called “the minor”.

177.5 Suppose someone said to you, “Three and a root of twenty: how much is its root?” This is the fifth binomial. We take its root as before, to get a root of five and a root of two and a half and a fourth, taking its root, and a root of five less a root of two and a half and a fourth, taking its root. It is called “[the number whose] power is a rational and a medial”. Its apotome is the fifth apotome, and an apotome of its root is a root of its apotome. It is called “the joining with a rational to become a whole medial”.

177.11 Suppose someone said, “A root of ten and a root of eleven: how much is its root?” This is the sixth binomial. We take its root as before, to get a half and a root of two and three fourths, taking its root, and a root of two and three fourths less a half, taking its root. It is called “[the number whose] power is a bimedial”. Its apotome is the sixth apotome, and an apotome of its root is a root of its apotome. It is called “the joining with a medial to become a whole medial”.

179.1 **Section Two, on adding and subtracting roots of numbers.**

179.2 **You multiply the two numbers, one of them by the other, when you want to add or to subtract their roots. If the result is a square, then the roots of the two numbers can be added and subtracted. If it is not a square, then they cannot be added or subtracted. If you know they can be added, take two roots of the result and add it to the sum of the two numbers. Take a root of this sum to get the required number.**

179.7 For example, suppose someone said, “Add a root of three to a root of twenty-seven”. We multiply the three by the twenty-seven, giving eighty-one, which is a square. We take two of its roots, giving eighteen. We add it to the sum of the two numbers, and a root of that sum is the required number, which is a root of forty-eight.

179.11 Another way is that we divide one of the two addends by the other, we add one to the result, and we multiply the sum by the divisor. The result is their sum.[96] So here we divide a root of the twenty-seven by a root of the three, resulting in three. We add one to it, and we multiply the sum by a root of the three, the divisor, as explained [in the section on] multiplying roots.[97] This results in a root of forty-eight, which is the required number, as shown above.

179.16 Another example: suppose someone said, “Add a root of two to a root of eight”. We multiply the two by the eight, giving sixteen, and we take two of their roots, giving eight. We add it to the sum of the two numbers, and a root of that sum is the required number, and that is a root of eighteen. If we wish, we can work it out the second way to get the required number.

179.20 Another example: suppose someone said, “Add half a root of twenty to two roots of five”. Half a root of twenty is less than one root, so we transform it to one root, as described in the section on division.[98] This gives, according to what was explained on working out the multiplication, a root of five. And two roots of five are more than one root, so we transform them to one root to again get a root of twenty. It is as if someone had said, “Add a root of five to a root of twenty”. We work it out as before to get the required number, which is a root of forty-five.

180.6 Likewise, whenever the roots are of different ranks, we transform them to one rank. For example, suppose the addend is a root of a number rational in square and the augend is a root of a root of a number, that is, a medial. We transform [the number] rational in square to a medial in ratio[99] with the other [addend], at which point we add them.[100]

180.10 Another example: suppose someone said, “Add a root of three to a root of fifteen”. We find that their surface is not a square, so they are incommensurable. We thus add them with the

[96] Judging by the corresponding passage at 181.10, this sentence should have been attributed to Ibn al-Bannāʾ. It is not in the *Condensed Book*, so it should be in SMALL CAPS.
[97] At 183.1 below.
[98] At 187.1 below.
[99] I.e., commensurable.
[100] Judging by the passage at 182.1, this passage should have been attributed to Ibn al-Bannāʾ. It is not in the

coordinating conjunction to get a root of three and a root of fifteen. Any examples like this, that can only be added with the coordinating conjunction, are called binomials.

180.15 Another example: suppose someone said, "Add half a root of a root of eighty to a third of a fourth of a root of six hundred eighty-four". It is known, according to what is explained in the section on multiplication,[101] that half a root of a root of eighty is a root of a root of five, and that a third of a fourth of a root of six hundred eighty-four is a root of four and three fourths. It is as if someone had said, "Add a root of a root of five to a root of four and three fourths". We transform them to the same rank, as mentioned before. So the problem becomes as if someone had said, "Add a root of a root of five to a root of a root of twenty-two and four eighths and half an eighth". Their surface is likewise not a square, so we add them with the coordinating conjunction, and that is a root of a root of five and a root of a root of twenty-two and four eighths and half an eighth. So know it.

181.4 **For subtraction, you subtract two of the roots that result from multiplying the two numbers from the sum of the two numbers, and you take a root of the remainder to get the required number.**

181.6 For example, suppose someone said, "Subtract a root of eight from a root of thirty-two". We multiply the eight by the thirty-two, giving two hundred fifty-six. We drop two of its roots, which are thirty-two, from the sum of the two numbers, and we take a root of the remainder. It is the required number, which is a root of eight.

181.10 ANOTHER WAY TO DO THIS IS TO DIVIDE ONE OF THE SUBTRAHENDS BY THE OTHER. THEN TAKE THE DIFFERENCE BETWEEN THE RESULT AND ONE, AND MULTIPLY BY THE DIVISOR TO GET THE REQUIRED NUMBER.

181.12 For example: suppose someone said, "Subtract a root of twelve from a root of twenty-seven". We divide a root of the twenty-seven by a root of the twelve, resulting in one and a half. We take the difference between it and the one, which is a half. We multiply it by a root of the twelve, which is the divisor, resulting in a root of three, which is the required number.

181.16 If we denominate a root of the twelve with a root of twenty-seven, it results in two thirds. We take the difference between it and the one, and that is a third. We multiply it by a root of the twenty-seven, the denominating number, and the result is the required number, which is a root of three. So know it.

182.1 IF THE SUBTRAHEND OR THE MINUEND IS MORE THAN ONE ROOT OR LESS, OR IF THE RANKS OF THEIR ROOTS ARE DIFFERENT, THEN IT IS NECESSARY TO TRANSFORM THEM TO ONE ROOT OR THE SAME RANK, AS WITH ADDITION.

182.4 Another example: suppose someone said, "Subtract a root of eight from a root of ten". We find that their surface is not a square, so they are incommensurable. So subtract it using the particle of exclusion, and that is a root of ten less a root of eight. Work out [other problems] similarly. So know it. Here, too, any examples like this one that can only be subtracted using the particle of exclusion are called apotomes. So understand it.

Condensed Book, so it should be in SMALL CAPS.

[101] At 186.11 below.

183.1 **Section Three, on multiplying roots.**

183.2 **To work this out you multiply one of the numbers by the other and you take a root of the result. This is the result of multiplying a root of one of them by a root of the other.**

183.4 For example, suppose someone said, "Multiply a root of eight by a root of nine". We multiply the eight by the nine, and a root of the outcome is the required number, which is a root of seventy-two.

183.7 Another example: suppose someone said, "Multiply a root of a root of five by a root of a root of seven". We multiply the five by the seven, and we drop a root of the root, which the two multiplicands had, on the result. This gives the required number, which is a root of a root of thirty-five. The rule is similar for other medials, no matter how far from rational they are.

183.11 Another example: suppose someone said, "Multiply a root of a root of a root of three by a root of a root of a root of eight". We multiply the eight by the three, and we drop a root of a root of the root on the result to get the required number, and that is a root of a root of a root of twenty-four.

183.15 Another example: suppose someone said, "Multiply three by a root of seven less two". To work this out we multiply the three by a root of the seven, and we subtract from the outcome its multiplication also by the excluded two. The remainder is the required number, which is a root of sixty-three less six. The principle behind multiplying appended and deleted terms will be covered in [the chapter on] algebra,[102] with the power of almighty God.

183.20 **If you want to multiply a number by a root of a number, square the number and work with the two numbers as shown above.**

184.1 For example, suppose someone said, "Multiply three by a root of seven". We multiply the three by itself, giving nine. Then it is as if someone had said, multiply a root of nine by a root of seven. We work it out as before, resulting in the required number, which is a root of sixty-three.

184.4 Another example: suppose someone said, "Multiply two by a root of a root of three". We multiply the two by itself, and the result by itself. This is multiplied by the three. A root of a root of the outcome is the required number, which is a root of a root of forty-eight. The rule is similar if there are more [roots] than these. So know it.

184.8 From this principle, I mean his saying "if you want to multiply a number by a root of a number, etc.", one knows how to work it out by transforming the problem to one root when there is a term with more than one root or less than one root, and by duplicating a root of a number or partitioning it.

184.11 Here is an example with a root that is more than one root. Suppose someone said, "Multiply two by two roots of seven". It is necessary that we focus on what [makes] this number of a root of the seven a [single] root. To do this we multiply the two, which is the number

[102] At 228.9 below.

of the root, by itself, and the outcome by the seven. We take a root of the result, which is the required number, and that is a root of twenty-eight. So it is as if someone had said, “Multiply two by a root of twenty-eight”. We work it out as before, resulting in the required number, which is a root of one hundred twelve.

184.18 Another example: suppose someone said, “Multiply five by three roots of a root of two”. So we likewise focus on what number makes three roots of a root of two a [single] root of a root. To do this we multiply the three, the number of roots, by itself, and the outcome by itself, and the outcome of that by the two, and we take a root of a root of the result. This is the required number, which is a root of a root of one hundred sixty-two. So it is as if someone had said, “Multiply five by a root of a root of one hundred sixty-two”. We work it out as before to get the required number, which is a root of a root of one hundred one thousand two hundred fifty. Work it out similarly if there are more [roots] than these.

185.6 Here is an example with less than one root: suppose someone said, “Multiply two thirds by half a root of twenty”. So we focus on half a root of twenty, and we change it into what [single] root it is, as before. To do this we multiply the half by a root of the twenty, so it gives, as explained, a root of the five. It is then as if someone had said, “Multiply two thirds by a root of five”. We again work it out as before. The result is the required number, which is a root of two and two ninths.

185.12 Another example: suppose someone said, “Multiply a root of five by half a root of a root of forty”. We focus on half a root of a root of forty, and we change it into what root of a root it is, as before. To do this we multiply the half by itself, and the outcome by itself, and the outcome of that by the forty. We get, as we have explained, a root of a root of two and a half. It is then as if someone had said, “Multiply a root of five by a root of a root of two and a half”. We work it out again as before, resulting in the required number, which is a root of a root of sixty-two and a half. Work out similar [examples] the same way.

186.1 Here is an example of duplicating roots: suppose someone said, “Duplicate a root of three twice”. It is as if someone had said, “Multiply two by a root of three”. Work it out as before to get the required number, which is a root of twelve.

186.4 Another example: suppose someone said, “Duplicate a root of seven five times”. It is as if someone had said, “Multiply five by a root of seven”. Work it out as before to get the required number, which is a root of one hundred seventy-five. Work out similar [examples] the same way.

186.8 An example of partitioning roots: suppose someone said, “How much is half of a root of ten?” It is as if someone had said, “Multiply a half by a root of ten”. Work it out as before to get the required number, which is a root of two and a half.

186.11 Another example: suppose someone said, “How much is a third of four eighths of a root of a root of sixty?” It is as if someone had said, “Multiply a third of four eighths by a root of a root of sixty. Work it out as before to get the required number, which is a root of a root of two sixths of a ninth and half a sixth of a ninth. So know it.

187.1 **Section Four, on dividing and denominating roots of numbers.**

187.2 **You divide the number by the number or you denominate it with it, and you take a root of the result. What you get is the result of dividing a root of the dividend by a root of the divisor.**

187.4 For example, suppose someone said, “Divide a root of twenty by a root of three”. We divide the twenty by the three, and we drop the root on the result to get the required number, which is a root of six and two thirds.

187.7 Another example: suppose someone said, “Divide a root of three by a root of eight”. We denominate the three with the eight, and we drop the root on the result to get the required number, which is a root of three eighths.

187.10 Work it out similarly for medials. For example, suppose someone said, “Divide a root of a root of six by a root of a root of two”. We divide the six by the two, and we drop a root of the root on the result to get the required number, which is a root of a root of three.

187.14 Another example: suppose someone said, “Divide a root of a root of eighteen by a root of a root of thirty-two”. We denominate the eighteen with the thirty-two, and we drop a root of the root on the result to get the required number. This is a root of a root of four eighths and half an eighth. The same rule applies to other medials, no matter how far from rational they are.

188.1 **In these [preceding] three sections,** ON ADDITION, MULTIPLICATION, AND DIVISION, **whenever you encounter a term that is more than one root or less than one root, or roots of different ranks, transform them to one root or make them the same rank.**

188.4 I say I have already introduced examples of these and how to work them out in the sections on addition and multiplication, but let us also present some examples in this section.

188.6 Along these lines, suppose someone said, “Divide a root of a root of fourteen by a root of two”. The divisor is rational in square and the dividend is medial. So we transform a root of the two so that it is medial like the divisor, and then we divide. We get a root of a root of four. If we divide the dividend by it and we drop a root of the root which the two dividends had on the result, it gives the required number, and that is a root of a root of three and a half.

188.11 Another example: suppose someone said, “Divide two roots of fifteen by two”. We already knew that two roots of fifteen are the same as a root of sixty, and that the two is a root of four. So it is as if someone had said, “Divide a root of sixty by a root of four”. We work it out as before to get the required number, which is a root of fifteen.

188.15 Another example: suppose someone said, “Divide half a root of twenty-four by a root of two”. We already knew that half a root of twenty-four is the same as a root of six. So it is as if someone had said, “Divide a root of six by a root of two”. We work it out as before to get the required number, which is a root of three.

188.18 **For division by binomials and apotomes, you multiply the dividend and the divisor by an apotome of the divisor if it is a binomial, or by its binomial if it is an apotome. Then you divide the result of the dividend by the result of the divisor.**

189.1 For example, suppose someone said, "Divide twelve by five and a root of three". We multiply the twelve, the dividend, by the five and a root of three, an apotome of the divisor, and we distribute across the appended and the deleted terms, as will be made clear in [the chapter on] algebra,[103] to get sixty less a root of four hundred thirty-two. This is the result for the dividend. Then we divide it by the result of multiplying the five and a root of three, the divisor, by the five less a root of three, its apotome, and that gives twenty-two, since any binomial multiplied by its apotome, or vice versa, results in the difference between the squares of the terms. The result from the division is the required number, which is two and eight parts of eleven parts less a root of nine parts of eleven and nine parts of eleven parts of eleven.

189.11 Another example: suppose someone said, "Divide ten by three less a root of seven". We multiply the ten, the dividend, by the three and a root of seven, the binomial of the divisor, to get thirty and a root of seven hundred. We divide it by the result of multiplying the three less a root of seven, the divisor, by its binomial, and that is two. The result of the division is the required number, which is fifteen and a root of one hundred seventy-five. So know it.

189.16 This completes Chapter Three, with the blessing and help of God.

191.1 **Part Two, on the basic rules by which one arrives at knowledge of the required unknown from the posited known.**

191.4 **This is divided into two chapters: a chapter on working out [problems] with proportion, and a chapter on algebra.**

193.1 **Chapter One, on working out [problems] with proportion. This can be done in two ways, with four proportional numbers and with scales.**

195.2 I say there are different kinds of proportions. Among them are arithmetical, harmonic, harmony, and ex-aequali proportion. For arithmetical [proportion], its principles have already been covered,[104] and for the other three, they are not described.[105] As mentioned, they are dispensable, since they originate from it, and it is the foundation of calculation. All three derive from it and it does not derive from them, as was shown in *Lifting the Veil*.[106]

195.7 **Four proportional numbers are those for which the ratio of the first to the second is as the ratio of the third to the fourth, and the product of the first by the fourth equals the product of the second by the third.**

195.9 **Whenever you multiply the first by the fourth and you divide by the second, it results in the third; or by the third, it results in the second. And whenever you multiply the second by the third and you divide by the first, it results in the fourth; or by the fourth, it results in the first. Whichever is unknown can be worked out from the other three known numbers.**

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[103] As in the example at 228.11.
[104] At 73.17 above.
[105] I.e., in Ibn al-Bannāʾ's *Condensed Book*.
[106] (Ibn al-Bannāʾ 1994, 293.17). By "as mentioned" he means "as mentioned in *Lifting the Veil*". See our commentary at 195.2.

195.14 **The way to do this is that you multiply the isolated number different in kind from the others by the number in ratio with the unknown, and then you divide by the third number, resulting in the unknown.**

195.16 For example, the ratio of three to six is as the ratio of four to eight. So the three with respect to the six is a half, and the four with respect to the eight is a half. The product of the first, which is the three, by the fourth, which is the eight, equals the product of the six, which is the second, by the four, which is the third.

195.19 If we multiply the three by the eight and we divide it by the six, it results in the four; or by the four, it results in the six. And similarly, if we multiply the six by the four and we divide by the three, it results in the eight; or by the eight, it results in the three.

196.3 Suppose in this example that the fourth is unknown, which is the eight, and we want to find it. So we work it out as mentioned. We multiply the isolated number – different in kind from the other two, namely in its classification, which is what is intended, which is discussed in this book,[107] and which in this example is six, since it is related to it, and thus the remaining two are related – by the number in ratio with the unknown, which is the four, since the fourth proportional is unknown. This gives twenty-four. We divide it by the third of the three remaining [numbers], which is the first, resulting in eight, which is the unknown fourth [number]. If the first is unknown, we divide the twenty-four by the eight to get the three.

196.10 If the second is unknown, which is the six, then we multiply the eight – which is the isolated number, since it is the one related to it, thus the remaining two are related – by the number in ratio with the unknown, which is the first, since the second is the unknown in this proportion, giving twenty-four. We divide it by the four, which is the third of the remaining three. The result is six, which is the unknown second. If the third is unknown, which is the four, we divide the twenty-four by the six, resulting in the four. So know it.

196.16 For the four proportional numbers, *if one switches to get the ratio of the first to the third and the second to the fourth; or reverses them to get the ratio of the second to the first and the fourth to the third; or combines to get the ratio of the sum of the first and the second to one of them is as the ratio of the sum of the third and the fourth to one of them; or separates to get the ratio of the difference between the first and the second to one of them is as the ratio of the difference between the third and the fourth to one of them; or combines after switching them; or separates after switching them; or switches after combining them; or switches after separating them; or reverses after any one of these, they will remain proportional.*[108]

197.4 There are four ways to find the unknown other than those mentioned. First, if the fourth, for example, is unknown, we divide the second by the first, and we multiply the result by the third, to get the fourth. Second, we divide the third by the first, and we multiply the result by the second, to get the fourth. Third, we divide the first by the second, and we divide the third by the result, to get the fourth. Fourth, we divide the first by the third, and we divide the second by the result, to get the fourth. These are not presented by the author because what he presented is the source of all of them. They derive from it, and it does

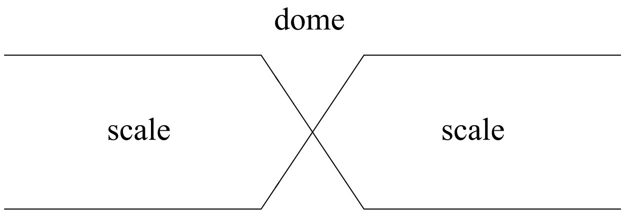
[107] Just above, at 195.14. Ibn al-Bannāʾ does not discuss it in *Lifting the Veil*.
[108] Copied from (Ibn al-Bannāʾ 1994, 294.3-9).

not derive from them. As the jurist Abū Muḥammad ʿAbd al-Ḥaqq ibn Ṭāhir reported, the first way is called the procedure, and the other four are called deductions by analogy.

198.2 **The [method of] scales comes from the art of geometry.** It comes from the art of geometry because ***the ratio of the error of each scale to the difference between the scale and the unknown number is as the ratio of the assigned number to the unknown.***[109]

198.4 You switch, you separate, you switch. The ratio of the portion to its scale is as the ratio of the assigned number to the unknown, as was made clear in *Lifting the Veil*.[110]

198.7 **You represent this by drawing a balance, as in this figure:**[111]



198.8 **You place the given assigned number above the dome. You choose any number you wish for one of the scales, and you perform the prescribed additions, reductions, and so forth from among the operations. Then you confront it with what is above the dome. If you got it right, then this scale is the unknown number. If you got it wrong, then write the error above the scale if it exceeds, or below if it falls short.**

198.13 **Then you choose for the other scale any number you wish other than the first [number], and you work it through as you did with the first [scale]. Then multiply the error of each scale by the posited number**[112] **in the other [scale]. Then look. If the two errors both exceed or both fall short, then subtract the smaller from the greater, and the smaller of the two products from the greater. Then divide the remainder from the two products by the remainder from the two errors. And if one of them exceeds and the other falls short, you divide the sum of the products by the sum of the errors,** resulting in the required number.

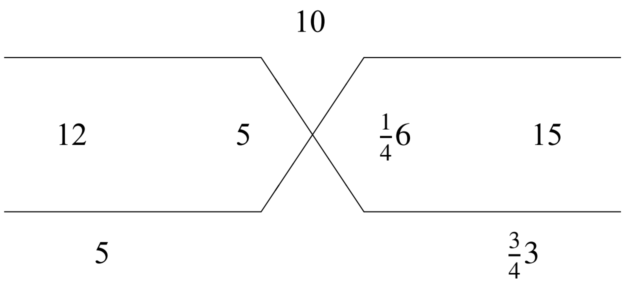
199.1 For example, suppose someone said, "A quantity: taking away its third and its fourth leaves ten. How much is the quantity?"

199.2 We draw the balance as described, and we place the given ten above the dome. Then we choose fifteen for the first scale. We place it between the lines of the balance. Then we take its third and its fourth, which are eight and three fourths, and we drop it from it, leaving six and a fourth, which is the portion that we confront with what is above the dome. So we place it inside the balance, next to the scale. The remainder from the ten above the dome,

[109] Copied from (Ibn al-Bannāʾ 1994, 297.17-18).
[110] (Ibn al-Bannāʾ 1994, 301.4ff).
[111] This is the only figure in Ibn al-Bannāʾ's *Condensed Book*.
[112] Literally "the whole [number]", i.e., the number in its entirety before performing operations and taking the difference with the number above the dome. We also translate this as "posited number" in the passages

after the confrontation, is three and three fourths. This is the error for the scale, which falls short. We place it below the scale as mentioned. If the portion confronted with it were, for instance, ten, then the scale would be the unknown.

199.9 We choose twelve for the second scale. We also place it inside the balance, on the other side, and we take its third and its fourth, giving seven. We subtract it from it, leaving five, which is also the portion we confront it with. We place it inside the balance, next to the scale. The remainder from the ten after the confrontation is likewise five, which is the error for the second scale, and it falls short. We place it below the scale, as in this figure:

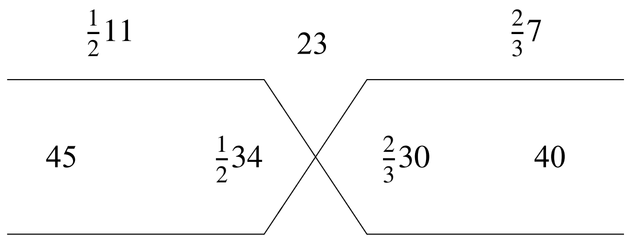


199.14 Then we multiply the three and three fourths, the first error, by the twelve, the second posited number, to get forty-five, which is one of the two products. Then we multiply the five, the second error, by the fifteen, the first posited number, to get seventy-five, which is the second product. From this we drop the first product, which is smaller, since the two errors fall short, leaving thirty. We keep it in mind. Then we drop the first error, since it is smaller, from the second error, since it is greater. The remainder is one and a fourth. We divide the remembered number by it. The result is the unknown quantity, which is twenty-four.

200.1 Another example: suppose someone said, “A quantity: we take the sum of its third and its fifth, and we add to it half of the remainder, so it comes to twenty-three. How much is the quantity?”

200.3 We draw the balance similarly and we place the twenty-three above its dome. We choose forty for one of the scales. We take its third and its fifth, and we add half of the remainder to it, as mentioned, to get thirty and two-thirds. This is the portion we confront with what is above the dome. The error is seven and two-thirds, which is an excess. We choose forty-five for the second scale. We take its third and its fifth and half of the remainder to get thirty-four and a half, which is the portion we also confront. The error is eleven and a half, which exceeds. So we place it and the first [error] above the scales, to get this figure:

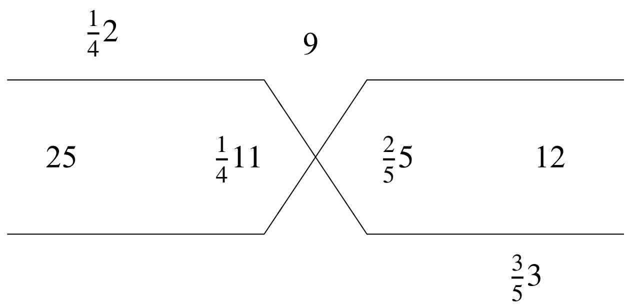
at 199.14, 200.10, 201.10, and 201.14.



200.10 Then we multiply the seven and two-thirds, the error of the first scale, by the forty-five, the second posited number, to get three hundred forty-five, which is the first product. Doing the reverse[113] gives four hundred sixty, which is the second product. From this we drop the first product, which is the smaller, since the two errors exceed. The remainder is one hundred fifteen. We divide it by the difference between the two errors, which is three and five sixths. The result is the unknown quantity, which is thirty.

201.1 Another example: suppose someone said, “A quantity: we add its tenth to the difference between its fourth and three of its fifths, so it comes to nine. How much is the quantity?”

201.3 We draw the balance similarly, and we choose twelve for one of the scales, and we drop its fourth from three of its fifths. The remainder is four and a fifth. To this we add a tenth of the scale, to get the sum five and two-fifths, which is the portion that we confront with what is above the dome. The error is three and three fifths, falling short. We put it below the scale. Then we choose twenty-five for the second scale. We again drop its fourth from three of its fifths, leaving eight and three fourths. To this we add a tenth of the scale to get the sum eleven and a fourth, which is the portion that we also confront. The error is then two and a fourth, which exceeds. We put it above the scale to get this figure:



201.10 Then we multiply the first error by the second posited number, giving ninety. Doing the reverse gives twenty-seven. We add these products, since one of the errors exceeds and the other falls short, to get the sum one hundred seventeen. We divide it by the sum of the two errors, which is five and four fifths and a fourth of a fifth. The result is the unknown quantity, which is twenty. So know it.

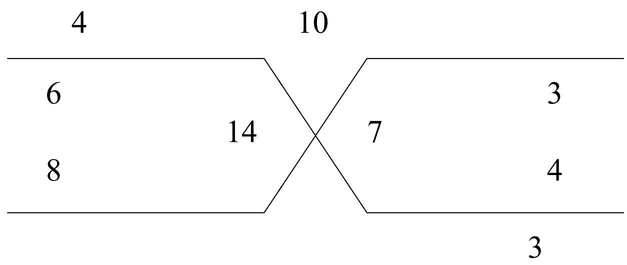
201.14 **If you wish, you can choose the first number or any other number for the second scale, then figure its portion, which you confront with what is above the dome. Multiply it by the first posited number, and multiply the first error by the second posited number. If the first error falls short, you add the two products. If it exceeds, you take**

[113] That is, the same calculation with the scales reversed.

their difference. You divide the result by the portion of the second scale to get the required number.

202.3 This second method is used only when there is a proportional relation. For example, suppose someone said, “Ten: you divide it into two parts so that a third of one of them is a fourth of the other”.[114]

202.5 We draw the balance similarly, and we put the ten above the dome. Then we choose two numbers for the scale so that a third of one of them is a fourth of the other. They are, for example, three and four. We confront their sum, since it is the portion, with the ten. The error is three, which falls short. Then we choose [numbers for] the second scale the same way, such as six and eight. We figure its portion, which is the sum of the numbers, to get fourteen. This gives the following figure:

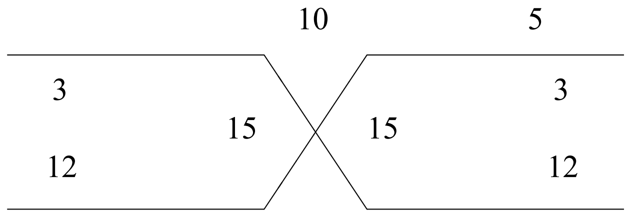


202.10 Regardless of which part we want to find, we multiply the first error by the upper number of the second, and the portion of the second by the upper number of the first. Then we continue with the procedure. The result is the required part, and the remainder from the ten is the other part. Suppose in this example that we want the smaller part. We multiply the three, the first error, by the six, the upper number of the second, to get eighteen. Then we multiply the fourteen, the second portion, by the three, the upper number of the first, to get forty-two. We add it with the first product, since the error falls short, to get sixty. We divide it by the fourteen, the second portion. The result is the required number, which is four and two-sevenths. The remainder from the ten is the greater part, which is five and five sevenths.

203.1 Another example: suppose someone said, “Ten: you divide it into two parts. You divide the greater by the smaller, so the result is four. How much is one of them?”

203.3 One of the parts is necessarily four times the other. So we draw the balance similarly, and we place the ten above the dome. Then we choose for one scale two numbers so that one of them is a fourth of the other. They are, for example, three and twelve. We confront their sum with the ten, since it is the portion. The error is five, which exceeds. Then we make our choice for the second scale. Here we choose the same as the first. We figure its portion, which is fifteen. This gives the following figure:

[114] Copied from (Saidan 1986, 557.7).



203.8 We work it out as described in the previous example, to find that one of the parts is two, and the other is the remainder from the ten, which is eight. So know it. If we wish, we can work out these two problems by the first method.

203.11 We follow the second method *in any problem where the assigned numbers are one of the scales and its error. For example, I say, "A quantity: we subtract its third and its fourth from a third of sixty and its fourth, leaving fourteen. How much is the quantity?"*

203.15 *The sixty is one of the scales, and the fourteen is its error, which exceeds. We choose whatever number we wish for the other scale, and we find its third and its fourth, which is the portion that we would confront with what is above the dome, if there were a number there. We work it out as mentioned, so the quantity is thirty-six.*

204.1 *Thus, the method of scales derives from the four proportional numbers. You know that approaches for working it out come from relating the two errors and from the differences of the two scales when combined and separated.*[115]

204.3 Regarding this, my professor, the jurist and great erudite Abū l-ʿAbbās [Ibn al-Bannāʾ], God bless him, dictated to me while I was studying with him on Wednesday, the twenty-eighth of the month of *rajab* in this year:

204.6 "There are three approaches. One of them is that you multiply the difference between the scales by one of the errors. If the two errors both exceed or both fall short, you divide the product by their difference. If one of them exceeds and the other falls short, you divide the product by their sum. You add the result of this to the scale whose error you multiplied if it falls short, and you subtract it from it if it exceeds. This gives the required number.

204.12 "The second approach is that you multiply the difference between the scales by the sum of the errors if they both exceed or both fall short, and you divide by their difference. If one of them exceeds and the other falls short, you multiply the difference between the scales by the difference between the errors, and you divide by the sum of the errors. Keep this result in mind. If you wish, you add this remembered number to the difference between the scales and you take half of the sum. You add it to the scale whose error is greater if it falls short, and you subtract it from it if it exceeds, to get the required number.

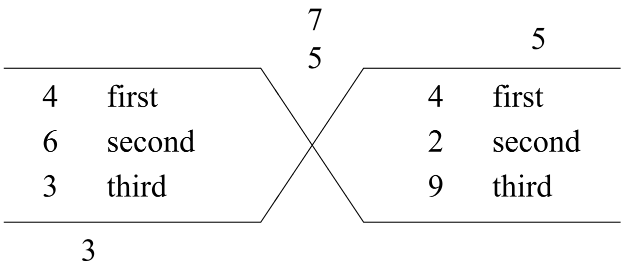
204.18 "And, if you wish, take the remembered number and the difference between the scales, subtract the smaller from the greater, and take half of the remainder. Add it to the scale whose error is smaller if it falls short, and subtract it from it if it exceeds, to get the required number.

[115] Copied from (Ibn al-Bannāʾ 1994, 298.13-19).

205.1 "The third approach is that you multiply the difference between the scales by your assigned number. If the errors both exceed or both fall short, you divide the product by their difference, and if one of them exceeds and the other falls short, you divide the product by their sum, to get the required number. So know it."

205.5 *Scales can also be set up to find the unknown in cases without a proportional relation, as in this problem: Three men want to buy a horse.*[116] *The first says to the second, give me half of what you have, and, together with what I have, I will have the price of the horse. The second says to the third, give me a third of what you have, and, together with what I have, I will have the price of the horse. And the third says to the first, give me a fourth of what you have, and, together with what I have, I will have the price of the horse.*

205.11 *We choose one scale for the three men. We assign to the first man whatever we wish, so we make it four. The second also has whatever we wish, so we make it two. Then the price of the horse is calculated to be five. We put it above the dome, and it is what we will confront. The third man has, also by calculation, nine. If we add a fourth of what the first has, it amounts to ten. The error of the first scale with these three numbers is five, which exceeds. Then we choose [the numbers in] the other scale. We assign four for the first man, the same as we assigned for him in the first [scale], and we make the second whatever we wish. We see that it should not be eight or more*, since it would lead to the third having nothing. *So know it. So we make it six. This makes the price of the horse, which we confront, seven. We put it above the dome also. The third necessarily has three, and adding this to a fourth of what the first has gives four. The error of the second scale is then three, which falls short.* This is the figure:



206.1 *We multiply the error of each scale by what each one has in hand in the other scale, and we divide the sum of the two products as mentioned, to get what each one has in hand and the price of the horse.*[117] This gives us what the first has in hand, which is the four, the second four and a half, the third five and a fourth, and the price of the horse is six and a fourth.

206.5 If we wish the result to be without fractions, we multiply all of the problem by the smallest number divisible by their denominators. We knew [how to do] this before,[118] and it is four. By this calculation the first has sixteen, the second has eighteen, the third has twenty-one, and the price of the horse is twenty-five. So understand it.

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[116] The word *dābba* means a "riding animal", such as a horse, mule, or donkey. We translate it as "horse".
[117] Copied from (Ibn al-Bannāʾ 1994, 299.1-14).
[118] At 122.2 above.

206.9 *If we wish, we can change what is assigned for the first [man] in the first scale, and leave unchanged what is assigned for the second, since the condition is that one of them should have the same number in both scales.*

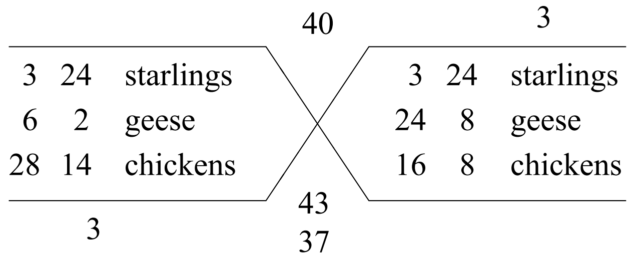
207.1 *If we [choose to] assign the price of the horse, we put it above the dome, and we put some portion of it for the first [man], and twice its remainder for the second. Then we drop a fourth of what we put for the first from the price of the horse, leaving what the third has. Then we take what the second has and a third of what the third has, and we confront it with the assigned price. We then do this likewise for the other scale: we first make the price of the horse whatever we wish, as long as it is not the first number. Then we work it out as before.*

207.6 *Another problem: forty birds, among which are geese, chickens, and starlings, [all] for forty dirhams. The starlings are eight for a dirham, the chickens are one for two dirhams, and the geese are one for three dirhams. How many were chosen of each kind of bird?*

207.9 *Not all examples of this type can be worked out. Two conditions must be satisfied. One is that the numbers must be whole numbers and not fractions. Second, the price of one of the cheapest [birds], if multiplied by the number of birds, must be smaller than the total price; and the price of one of the most expensive [birds], if multiplied likewise, must be greater than the [total] price.*

207.13 *It is obvious in this problem that the number of starlings must be eight or sixteen or twenty-four or thirty-two, and no other [number]. If there are eight, then there remain thirty-two birds and thirty-nine dirhams. If we check the second condition, the product of the remaining individual birds by the price of the cheapest one is greater than the number of the [total] price, which is not valid. If we make the starlings sixteen, and we again check the remaining birds, then the remaining price is also not valid.*

208.1 *If we make the starlings twenty-four, and we again check what remains, the two conditions are met. So we suppose there are twenty-four starlings, and we assign the chickens to be whatever we wish. Suppose they are eight. Then the geese are eight, the remaining number. The error in the price is three dirhams, which exceeds. Then we choose [the numbers in] the other scale. We make the starlings twenty-four, as in the first [scale], since it is a condition for it to work out that a number be repeated in the two scales. And we make the chickens be whatever we wish that is different from the first, so let them be fourteen. Then the number of geese is two. The error is three dirhams falling short.* Here is the figure:



[119] Copied from (Ibn al-Bannāʾ 1994, 299.15-300.17).

208.8 *We work it out as before to get the required number of each kind of bird and the price of each kind, whichever we want to find first.*[119]

208.10 The price of the starlings is three, and their number is twenty-four. The geese are five and their price is fifteen, and the chickens are eleven and their price is twenty-two.

208.12 *If we make the starlings thirty-two, it is not valid since the condition for the remaining [birds] is not met. Thus this problem has only one answer. For similar problems, look to these two problems. Problems like this cannot be worked out by the second method of solving by scales, since that is specific to a proportional relation, as said above. Problems involving multiplication are not based in proportion, so they cannot be solved by the scales. So know it.*[120]

208.17 Chapter One is completed, with the praise of God and His help.

209.1 **Chapter Two, on algebra. We present this work in five sections.**

211.1 **Section One, on the meaning of algebra (restoration and confrontation) and an explanation of its types.**[121]

211.2 **Restoration is the reconstitution we mentioned in Part One of this book.**[122] **Confrontation is subtracting each species from like [species] until there are no longer two species of the same kind in the two parts.**[123] **And equalization is the restoration of the deleted to the appended, and the subtraction of the appended from the appended and the deleted from the deleted of things of the same kind.** This will be covered with clear examples in the section on addition and subtraction,[124] almighty God willing.

211.8 **Algebra depends upon three species: the number, the things, and the *māl*s. The things are roots,** since every unknown in numbers is a thing, and a root of its square when it is known. **The *māl* is what one gets from multiplying the root by itself.** The term *māl* serves to distinguish it from the other [species].

211.13 **These three species are equated to one another, in either a simple or composite way. This results in six types, three simple and three composite.**

211.15 **The first of the simple [types], following convention, is: some *māl*s equal some roots.** For example, three *māl*s equal seven things. **The second is: some *māl*s equal a number.** For example, five *māl*s equal twenty. **The third is: some roots equal a number.** For example, three roots equal twelve.

212.6 **In the first of the three composite [types], which is the fourth type, the number is isolated.** For example, a *māl* and ten roots equal twenty-four. **In the fifth, the roots are isolated.** For example, a *māl* and four equal five roots. **And in the sixth, the *māl* is isolated.** For example, a *māl* equals four roots and five.

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120 Copied from (Ibn al-Bannāʾ 1994, 300.18-22).
121 The "types" are the six types of equations, described beginning at 211.13.
122 At 129.1 for whole numbers, and at 154.1 for fractions.
123 I.e., in the two parts of a subtraction, or in the two sides of an equation.
124 At 219.1 below.

213.1 **Section Two, on working out the six types.**

213.2 **For the three simple [types], you divide by the *māl*s what they are equal to, and by the roots if there are none. The result of the division in the first and third types is the root, and in the second it is the *māl*. If the root is known, then the *māl* is known by multiplying the root by itself. And if the *māl* is known, then the root is known from it.**

213.7 For example, suppose someone said, "Three *māl*s equal fifteen things". The meaning of this problem is: what quantity (*māl*), if we take its root fifteen times, gives a sum equal to three times the quantity? This is of the first type, and it is worked out as explained above. If we divide the fifteen, the number of things, by the three, the number of *māl*s, it results in five, which is a root of the unknown *māl*, the latter being twenty-five.

213.13 Another example: suppose someone said, "Two *māl*s equal eighteen". The meaning of this problem, likewise, is: what quantity (*māl*), if we add itself to it, becomes equal to eighteen? This is of the second type. To work it out we divide the eighteen by the two, the number of *māl*s, as mentioned, to get nine. This is the unknown *māl*, and its root is three.

214.1 Another example: suppose someone said, "Five things equal twenty". The meaning of this problem, likewise, is: what quantity (*māl*), if we take five of its roots, gives a result equal to the twenty? This is of the third type. To work it out we divide the twenty by the five, the number of things, since there are no *māl*s. It results in four, which is a root of the unknown *māl*, and that is sixteen.

214.7 **To work out the fourth type, you halve the number of roots, you square the half, you add it to the number, you take a root of the result, and you drop the half from it, leaving the root.**

214.9 For example, suppose someone said, "A *māl* and two things equal fifteen". Its meaning is, what quantity (*māl*), if we add two of its roots to it, becomes equal to fifteen? We take half of the things, giving one, and we square it, giving one, and we add the number to it, giving sixteen. We take its root, giving four. From this we drop the one, which is the half, leaving three. This is the unknown thing, and the *māl* is nine.

214.14 If we wish to find the *māl* first, before the root, we add half a square of the number of roots to the number, then keep the sum in mind. Then we subtract a square of the number from a square of the remembered number, and we take a root of the remainder. We subtract this from the remembered number. The remainder is the *māl*.

214.17 If we want to work it out in the example, we add half a square of the number of things, which is two, to the fifteen, giving seventeen. We keep it in mind. Then we square the number to get two hundred twenty-five. We subtract it from a square of the remembered number, which is two hundred eighty-nine, leaving sixty-four. We take its root, giving eight. We subtract it from the seventeen, leaving nine, which is the required *māl*.

215.3 **For the fifth [type], you subtract the number from a square of half of the number of roots, and you take a root of the remainder. If you add it to the half, it gives a root of the greater *māl*, and if you subtract it, it gives a root of the smaller *māl*.**

215.6 For example, suppose someone said, "A *māl* and eight equal six things". Its meaning is, what quantity (*māl*), if we add eight to it, gives a result equal to six of its roots? We take half of the things and we square it, giving nine. We drop the number from it, leaving one. We take its root, giving one. If we add it to the half, it gives four, which is a root of the greater *māl*, so the greater *māl* is sixteen. If we subtract it from it, it leaves two, which is a root of the smaller *māl*, and the *māl* is four.

215.12 KNOW THAT **whenever a square of the half is equal to the number, then the half is the root, and the *māl* is the number.**

215.14 For example, suppose someone said, "A *māl* and nine equal six things". We square half of the things, giving nine, and this is equal to the number. So the number is the *māl*, and the half is the root. And if we continue the solution, we subtract the number from the square, leaving nothing. We take its root, giving nothing. We add nothing to the half, or we subtract it from it, leaving the half. It is the root, and its square is the number. So know it.

216.1 If we wish to find the *māl* first, before the root, we subtract the number from half of a square of the number of roots, and we keep the remainder in mind. Then we subtract a square of the number from a square of the remembered number. If we add a root of the remainder to the remembered number, it gives the greater *māl*; and if we subtract it from it, then the remainder is the smaller *māl*. Note that this is what it is when the number is smaller than a square of half of the number of roots.

216.6 If we want to work this out in the preceding example, we subtract the number from half of a square of the number of roots, which is eighteen, leaving ten. We keep it in mind. Then we subtract a square of the number, which is sixty-four, from a square of the remembered number. The remainder is thirty-six. We take its root, giving six. If we add it to the remembered number, we get sixteen, the greater *māl*. And if we subtract it from the remembered number, then the remainder is four, the smaller *māl*. So know it.

216.11 **The sixth [type] is solved like the fourth, except that you add the half at the end to a root of the sum to get the root.**[125]

216.13 For example, suppose someone said, "A *māl* equals two of its roots and three". Its meaning is, what quantity (*māl*) is equal to two of its roots and three? So we square half of the things and we add it to the number, giving four. We take its root, giving two. We add the half to it to get three, which is the unknown thing, and the *māl* is nine.

216.17 If we wish to find the *māl* first, before the root, we add double the number to a square of the roots, and we keep half of the sum in mind. Then we drop a square of the number from a square of the remembered number, and we add a root of the remainder to the remembered number. This gives the *māl*.

216.20 If we want to work it out in the example, we add double the number, which is six, to a square of the roots, giving ten. We keep in mind its half, which is five. Then we drop a square of the number from a square of the remembered number. The remainder is sixteen.

[125] In the *Condensed Book* this is placed just after the passage at 214.7.

We add its root, which is four, to the remembered number. This gives the unknown *māl*, which is nine.

217.1 **For each of the three composite types, whenever there is more than one *māl*, reduce it to one *māl*, and reduce by that term all of the equation. And for each of them, whenever there is less than one *māl*, restore it to one *māl*, and restore by that term all of the equation. The way to work out restoration and reduction was described earlier.**[126] **And if you wish, divide [each of] the names**[127] **in the problem by the number of *māl*s. This reduces the problem.**[128] **Then confront the two sides.**

217.8 He indicated this only for the composite [equations], since the simple [equations] do not require restoration if there is less than one *māl*, nor reduction if there is more than one *māl*.

217.10 For example, suppose someone said, "Two *māl*s and six things equal thirty-six". We reduce the two *māl*s to one *māl* by multiplying them by a half, as mentioned above. And we reduce by this [multiplication] the things and the number. So the problem becomes: a *māl* and three things equal eighteen, which is the fourth type. This is also how to work out the fifth and sixth [types].

217.15 And if we work it out the other way, we divide the two, the number of *māl*s, by itself, resulting in one. And we also divide the number of things by it, resulting in three. And we also divide the number by it. So the problem becomes: a *māl* and three things equal eighteen, just as before.

218.1 Another example: suppose someone said, "Half a *māl* and two things equal six". We restore the half to one *māl* by multiplying it by two. We multiply it also by the things and the number. So the problem becomes: a *māl* and four things equal twelve, which is also the fourth type. This is also how to work out the fifth and sixth [types].

218.6 And if we work it out the other way, we divide the half by itself, resulting in one. And we also divide the number of things by it, resulting in four. We also divide the number by it, resulting in twelve. So the problem becomes: one *māl* and four things equal twelve, just as before. So understand it.

219.1 **Section Three, on addition and subtraction.**

219.2 **Adding different species is done with the conjunction "and",** like: a *māl* and six things and ten dirhams.[129]

219.4 **…and the different excluded [species] cannot be subtracted.**

219.5 *For example, suppose someone said, "Add a* māl *less a thing to ten dirhams". The sum is: a* māl *and ten dirhams less a thing. So the exclusion remains as it was, since [the thing] cannot be subtracted from anything.*[130]

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[126] At 129.1 and 154.1 above.
[127] By "names" (here *alqāb*) he means the terms of the equation.
[128] Literally, "what results is a returning to the problem".
[129] Copied from (Ibn al-Bannāʾ 1994, 313.3).
[130] Copied from (Ibn al-Bannāʾ 1994, 313.3-5).

219.7 **Whenever appropriate, subtract the smaller from the greater, that is, in the problem.** *For example, suppose someone said, "Add two* māls *less a* māl *to ten dirhams". So we subtract the excluded* māl *from the two* māls*, so the remaining sum is a* māl *and ten dirhams.*

219.10 *Another example: suppose someone said, "Add a* māl *less two things to ten things". We subtract the two things from the ten things. The remaining sum is a* māl *and eight things.*[131]

219.12 *Another example: suppose someone said, "Add a* māl *less five things to ten things". We subtract the five things from the ten things. The remaining sum is a* māl *and five things.*[132]

220.1 **Subtracting different species is done using the particle of exclusion.** For example, suppose someone said, "Subtract a thing from a *māl*", then the remainder is a *māl* less a thing.

220.3 Another example: suppose someone said, "Subtract ten dirhams from two *māl*s and three things". The remainder is two *māl*s and three things less ten dirhams.

220.5 **The exclusion can be in both parts**[133] **or in one of them, and it might be one species or two different species. To work it out, you add the exclusion in each part to both parts at once, and then you subtract. Work out the two sides of an equation similarly whenever they have exceptions.**

220.9 For example, suppose someone said, "Subtract two and a thing from a *māl* less three things". We find the things on the side of the minuend to be deleted, and they are three. We add them to the two parts at once. This is what is meant by equalization with restoration, since we added to the *māl* what it is associated with, which are the three things. Thus we restore it from deleted to appended. It becomes greater, as is desired for the minuend. We [also] equalize by restoring the subtrahend by the amount we added in the minuend, which is by the three things, since subtracting two numbers is the same as subtracting them after having added a number to both of them, or after subtracting a number from both of them. The problem is as if someone had said: "Subtract two and four things from a *māl*". Then we work it out as before. The remainder after that is the required number, which is a *māl* less four things and less two. So know it.

221.1 Another example of this: suppose someone said, "Subtract fifty-two dirhams less five things from two cubes and thirty dirhams". We find the things on the side of the subtrahend to be deleted, and they are five. We add them to the two sides at once, as before. The problem becomes as if someone had said: "Subtract fifty-two dirhams from two cubes and five things and thirty dirhams". Then we find dirhams on both sides. We remove the smaller quantity from the two sides at once. The problem becomes as if someone had said, "Subtract twenty-two dirhams from two cubes and five things".

221.8 This is confrontation with equalization, since when we removed the thirty from the subtrahend it becomes smaller than the original subtrahend. So we equalize, by which we remove from the minuend the same amount we removed from the subtrahend. It is worked out like

[131] Copied from (Ibn al-Bannāʾ 1994, 313.6-9).
[132] Copied from (Ibn al-Bannāʾ 1994, 313.10-11), except that Ibn al-Bannāʾ solves "add a *māl* less ten things to five things".
[133] That is, in the minuend or the subtrahend.

before. We work the remainder as before. What remains after that is the required amount, which is two cubes and five things less twenty-two dirhams.

221.13 Another example: suppose someone said, "Subtract twelve dirhams less four things from three *māl*s less two things". We find two deleted things in the minuend, and also four deleted things in the subtrahend. We add them. Since they are of the same species, they amount to six things. We add them to the two parts at once, or we restore both the subtrahend and the minuend by what is excluded from them, and we add their equals to the other [part], as before. The problem becomes as if someone had said, "Subtract twelve dirhams and two things from three *māl*s and four things". So we confront and subtract. The remainder after that is the required number, and that is three *māl*s and two things less twelve dirhams.

222.1 Another example: suppose someone said, "Subtract a cube less two *māl*s from thirty dirhams less four things". We restore and subtract. The remainder is the required number, which is thirty dirhams and two *māl*s less a cube and less four things.

223.1 Subsection concerning examples with the two sides of an equation.

223.2 Along these lines, suppose someone said, "A *māl* less three things equal two and a thing". We find the things on the side of the *māl* to be deleted, and they are three. We restore the *māl* by them, and we also add them to the two sides of the equation[134] as we did before, since whenever one adds equals to equals or subtracts equals from equals, the outcomes are equal. So the problem becomes: a *māl* equals two and four things, which is the sixth type.

223.7 Another example: suppose someone said, "A *māl* less three things equal twenty-four dirhams less five things". We restore and confront as before, so the problem becomes: a *māl* and two things equal twenty-four dirhams, which is the fourth type. If we wish, we can first subtract the three things which are deleted on the side of the *māl* from the five things deleted on the side of the number. This leaves two things deleted on the side of the number. And we finish working it out, again arriving at the fourth type.

223.14 Another example: suppose someone said, "A *māl* less ten dirhams equal a *māl* less two things and a half". We restore and confront, so the problem becomes: ten dirhams equal two things and half a thing, which is the third type.

223.17 Another example: suppose someone said, "A *māl* and ten dirhams equal fifty-one dirhams less four things". We restore, and the problem becomes: a *māl* and four things equal forty-one dirhams, which is the fourth type.

224.1 Another example: suppose someone said, "A *māl* and five things equal ten dirhams and two *māl*s less a thing". We restore and confront, so the problem becomes: six things equal a *māl* and ten dirhams, which is the fifth type.

224.4 Another example: suppose someone said, "Two *māl*s and ten dirhams less two things equal three *māl*s and seven dirhams less seven things". We restore and confront, so the problem

[134] He should add them only to the side with two and a thing, because the act of restoration already adds them to the *māl*.

becomes: a *māl* equals three dirhams and five things, which is the sixth type. So know it. Work out similar problems the same way, with the help of God.

225.1 **Section Four, on multiplication and knowing the power and the term.**

225.2 **For the power, know that the power of the things is one, the power of the *māl*s is two, and the power of the cubes is three. As for the term, the term for one is "things", the term for two is "*māl*s", and the term for three is "cubes". After that, there are three for each cube and two for each *māl*.** THE CUBE IS THE SURFACE OF THE THING BY THE *MĀL*. IT IS CALLED THIS WAY BECAUSE IT IS A CUBE, ALTHOUGH YOU DO NOT KNOW ITS AMOUNT.

225.8 Suppose someone said, "What is the power of a *māl māl*?" We say four. And suppose someone said, "What is the power of a *māl* cube?" We say five. Suppose someone said, "What is the power of a *māl māl māl*?" We say six. And suppose someone said, "What is the power of a *māl* cube *māl* cube?" We say ten. To work these out, we always count each *māl* as two, since that is its power, as mentioned. Likewise, each cube is always three, since that is its power, as mentioned. The result is the associated power.

225.13 And suppose someone said, "What is the power of a *māl* cube *māl māl*?" We would say nine. Suppose someone said, "What is the power of a cube *māl* cube cube *māl māl*?" We would say fifteen. Work it out this way for other combinations, too.

226.1 Conversely, suppose someone said, "What is a term for four?" We would say a *māl māl*. And if someone said, "What is a term for seven?" We would say a cube *māl māl*. And if someone said, "What is a term for six?" We would say a *māl māl māl*, or a cube cube.

226.3 We can also work this out by separating the powers by twos or threes or by grouping them: first the *māl*, and then the cube, and then the *māl* and the cube. Suppose someone said, for example, "What is a term for eight?" We would say a *māl māl māl māl*. Or if we wish, we can say a cube *māl* cube, or a cube cube *māl*, or a *māl* cube cube, or anything else that is allowable.

226.7 Suppose someone said, "What is a term for nine?" We would say a cube cube cube, or if we wish, we can say a cube *māl māl māl*, or if we wish, we could continue or stop. [It can be] anything that is allowable. So know it.

226.9 **To multiply these species, add the power of the multiplicand and the power of the multiplier. Then the sum of the powers is the power of the result. To multiply a number by one of these species, the result is the same species.**

226.12 For example, suppose someone said, "Multiply five things by seven things". We multiply the number of the multiplicand, which is five, by the number of the multiplier, which is seven, giving thirty-five. Then we add the powers of the two multiplicands, giving two. This two is the power of the thirty-five resulting from the multiplication, so it is *māl*s. So the result of the multiplication is thirty-five *māl*s, which is the required number.

226.17 Another example: suppose someone said, "Multiply ten things by six *māl*s". We multiply the number of the multiplicand by the number of the multiplier, which gives sixty. Then we add the powers of the two multiplicands, giving three. We make it the power of the result, which you recall is sixty. This is the required number, which is sixty cubes.

227.4 Another example: suppose someone said, “Multiply a thing by a cube”. The result of multiplying the number of the multiplicand by the number of the multiplier is one. The sum of their powers is four, which is the power of the one, the result of the multiplication. So it is a *māl māl*.

227.7 Another example: suppose someone said, “Multiply six by four *māl*s”. We multiply the multiplicand by the number of the multiplier, giving twenty-four. The outcome of the multiplication in this example is *māl*s, so it is then twenty-four *māl*s.

227.10 Another example: suppose someone said, “Multiply seven by three *māl*s cube”. We multiply the multiplicand by the number of the multiplier, giving twenty-one. The outcome of the multiplication in this example is likewise *māl* cube, so it is then twenty-one *māl*s cube. So know it.

227.14 **Whenever you equate between the *māl*s *māl*s and the cubes and the *māl*s, or the cubes and the *māl*s and the things, or any combination in which you do not have a number, subtract the smaller of the powers from the power of each of them. Then you equate what remains, one with the others, in the same way as the equation.**

227.17 For example, suppose someone said, “Three *māl*s *māl* equal four cubes and ten *māl*s”. We find the power of the *māl*s to be the smallest power in the equation, and it is two. If we drop it from the power of the *māl*s *māl*s, which is four, there remains two, and its term is *māl*s. We also drop it from the power of the cubes, which is three, leaving one, and its term is a thing. And we reduce the *māl*s to a number. The problem becomes: three *māl*s equal four things and ten dirhams, which is the sixth type.

228.1 Another example: suppose someone said, “Three cubes equal ten *māl*s and twenty things”. We find the power of the things to be the smallest power in the equation, and we work it out as before. The problem becomes: three *māl*s equal ten things and twenty dirhams, which is also the sixth type.

228.4 Another example: suppose someone said, “A cube and ten *māl*s equal thirty-nine things”. We work it out as before. The problem reduces to: a *māl* and ten things equal thirty-nine dirhams, which is the fourth type. Work out similar examples the same way.

228.8 If it is not possible to transform it to [one of] the six types, then do not work it out since it would be of no use for anything.

228.9 **The multiplication of two appended or deleted [terms], one of them by the other, is appended, and the multiplication of an appended [term] by a deleted [term] is deleted.**

228.11 For example, suppose someone said, “Multiply five things by thirteen less four things”. We multiply the five things by the appended thirteen, and we drop from the result the product of the five things again by the deleted four things. The remainder, then, is the required number, which is sixty-five things less twenty *māl*s.

228.15 Another example: suppose someone said, “Multiply eight less two things by seven less four *māl*s”. We multiply the appended eight by the appended seven, to get an appended [amount]. We add to it the product of the two deleted things by the four deleted *māl*s, since it is appended. And we drop from the sum the product of the appended eight by the deleted four *māl*s, since it is deleted, and the product of the deleted two things by the appended seven, since it is deleted. The remainder, then, is the required [number], and that is eight cubes and fifty-six less fourteen things and less thirty-two *māl*s. So know it.

229.1 **Section Five, on division.**

229.2 **To divide one of these species by a lower species, drop the power of the divisor from the power of the dividend. What remains is the power of the species of the result of the division.**

229.4 For example, suppose someone said, “Divide ten *māl*s by two things”. We divide the number of *māl*s by the number of things, and we call the resulting five by the name for the difference between the power of the things and the *māl*s, and that is a thing. The result of the division is five things, which is the required number.

229.8 Another example: suppose someone said, “Divide fifteen cubes by three things”. We divide the number of cubes by the number of things, and we call the resulting five by the name for the difference between the power of the things and the cubes, and that is a *māl*. The resulting five is then five *māl*s, which is the required amount. So know it.

229.12 **And whenever you divide a species by itself, the result is a number.**

229.13 For example, suppose someone said, “Divide twelve *māl*s by three *māl*s”. We divide the number of the divisor by the number of the dividend, resulting in four. There is no difference between the powers,[135] so we call the term[136] of the resulting four “number”, most certainly.

229.16 **Whenever you divide one of these species by a number, the result is the same species.**

229.17 For example, suppose someone said, “Divide twelve things by four dirhams”. We divide the number of the dividend by the number of the divisor, resulting in three. And the number of the divisor has no power, so that dropping from the power of the dividend leaves the power of the dividend as the power of the result, so we call it with its term. The result is three things, which is the required number.

230.1 **If there is an exclusion in the dividend, divide both the excluded [term] and the diminished [term] by the divisor, and exclude the result of the excluded [term] from the result of the diminished [term]. The outcome of this is the result of the division.**

230.4 For example, suppose someone said, “Divide twelve cubes less three *māl*s by two things”. We divide the twelve cubes, which is the diminished [term], by the two things, which is the divisor. And the exclusion in this division is the three *māl*s, which is the excluded [term that we] likewise [divide] by the two things. The remainder is the required number, which is six *māl*s less a thing and half a thing.

[135] Literally, “there is nothing between the powers”.
[136] Literally, “we name the name”, with *ism*.

230.8 Another example: suppose someone said, "Divide ten *māl*s less three things by two dirhams". We work it out as before. The result is five *māl*s less a thing and half a thing, which is the required number.

230.11 **One cannot divide the lower of two species by the higher** EXCEPT BY CANCELLING THEIR COMMON DIVISOR, BY WHICH YOU SUBTRACT THE POWER OF THE SMALLER FROM THE POWERS OF BOTH OF THEM.

230.13 For example, suppose someone said, "Divide six *māl*s by three cubes". We divide the number by the number. Then the result is divided by the difference between the two ranks, which is a thing. The result is two divided by a thing.

230.16 **And one cannot divide by a diminished [term].** SO UNDERSTAND IT AND FIGURE IT OUT WITH THE HELP OF GOD.

230.17 For example, suppose someone said, "Divide ten *māl*s by three less a thing". So we say the result is ten *māl*s divided by three less a thing. The answer is the same as the question. So understand it.

231.1 SECTION [ON SECRET NUMBERS].

231.2 Let us conclude this work with three problems of witty reckoning, like those with which arithmeticians still conclude their compositions.

231.4 One of them is that we tell someone to drop his [secret] number from ten, then to drop a square of the remainder from a square of his number. If [a square of] the remainder is smaller, we ask for the remainder. We divide it by ten, and we add ten to the result and then take half of the sum[137] to get the secret number. And if a square of the remainder is greater, he drops a square of the secret number from it, and we ask for the remainder. We divide it by the ten, and we drop the result from ten. Half of the remainder is the secret number. If we wish, we can ask that he drop his secret number from something other than the ten. Following the procedure will give the required number.

231.13 The second problem. We tell someone to divide the ten into two parts secretly. Then we tell him to divide a square of one of them by their surface, and we ask for the outcome. If we knew this, it is the ratio of one of the two parts to the other. So we divide the ten according to this ratio. And similarly, we can work with whatever number we wish other than the ten, dividing it secretly. This results in the two secret parts.

232.1 The third problem. A secret number is divided into two secret numbers. How much is it? And how much are each of its parts? We tell him to multiply one of the parts by the other and to square each of them, then to drop a square of the smaller from a square of the surface, and we ask for the remainder; and to drop the surface from a square of the greater, and we ask for the remainder. We take a root of the difference between the two asked numbers, which is the difference between the parts. Then we divide the sum of the

[137] This sentence is our reconstruction of what appears to be a corrupt passage. Literally, it reads "We divide it by ten, and what results, we add to it half of its remainder to the ten". Only the Oxford manuscript attempts to correct it, by writing "We divide it by ten, and what results, we add to the remainder from the ten". This is mathematically correct, but we cannot add the remainder from the ten because that is not revealed to us. This MS also has "square" scratched in above "remainder".

asked numbers by it to get the secret number, which is the sum of the two parts. If we add to it a root of the difference between them, it gives double the greater, and if we subtract it from their sum, it leaves double the smaller. So know it.

232.9 These three problems were dictated to me by my professor, the great jurist Abū l-ʿAbbās [Ibn al-Bannāʾ], may God be pleased with him.

233.2 As the servant [of God], the named, the tempted, ʿAbd al-ʿAzīz ibn ʿAlī ibn Dāwud al-Hawārī al-Miṣrātī, may God have mercy on him, said: With the praise of God for His excellent assistance, I have achieved what I set out to accomplish to the extent of my ability, and not absolving myself of my errors and of anything leading to wrong ideas, I beg Almighty God's protection and to preserve me from loosing His favors. He is sufficient for us, and He is the best disposer of our affairs.

This work was completed on Saturday, the eighteenth of Dhū al-Qaʿda, 704. May God's generosity benefit the author. Full prayers upon His prophet and servant, our Master and Lord Muḥammad, and upon his family and companions and followers.