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Chapter 30
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single element or on a group of elements, such as those comprising the kth plane. Thus

$$w_{k,f}=\phi(f_k), \tag{3}$$

where f_k = component of force in the direction of the observed stress acting over the *whole* kth cross section. Similarly, for a backward unit dislocation,

$$w_{i,k,b}=\psi(f_{i,k}), \tag{4}$$

and

$$w_{k,b}=\psi(f_k). \tag{5}$$

In view of Eqs. (1) to (5) we have

$$\left.\begin{aligned}\phi(f_k)&=\prod_{i=1}^{N_k}\phi(f_{i,k}),\\ \text{and}\quad \psi(f_k)&=\prod_{i=1}^{N_k}\psi(f_{i,k}).\end{aligned}\right\} \tag{6}$$

But, clearly,

$$f_k=\sum_{i=1}^{N_k} f_{i,k}. \tag{7}$$

In accordance with a proof frequently cited in thermodynamics,[1] Eqs. (6) and (7) define the exponential relations:

$$\left.\begin{aligned}w_{k,f}&=\phi(f_k)=\exp c_1 f_k,\\ \text{and}\quad w_{k,b}&=\psi(f_k)=\exp c_2 f_k,\end{aligned}\right. \tag{8}$$

where c_1 and c_2 are parameters.

Assumption II. The probability of a unit dislocation per unit time in the direction of the force component (forward jump) is the same in either direction from the cross-sectional plane, i.e.,

$$\phi(f_k)=\psi(-f_k). \tag{9}$$

Now, since a forward jump and a backward jump are mutually exclusive events, the net probability of a forward jump per unit time is the difference $w_{k,f}-w_{k,b}$. The net strain in the direction of stress, occurring in time Δt, is evidently

$$\Delta\epsilon=n\lambda(w_{k,f}-w_{k,b})\Delta t. \tag{10}$$

From Eqs. (8) and (9) it follows that $c_1=-c_2$, and hence we have from Eqs. (8),

$$\left.\begin{aligned}w_{k,f}-w_{k,b}&=\exp c_1 f_k-\exp(-c_1 f_k)\\ &=2\sinh c_1 f_k.\end{aligned}\right\} \tag{11}$$

Thus,

$$\Delta\epsilon=n\lambda\times 2\sinh c_1 f_k\times\Delta t,$$

or

$$\lim_{\Delta t\to 0}\frac{\Delta\epsilon}{\Delta t}=\frac{d\epsilon}{dt}=2n\lambda\sinh c_1 f_k. \tag{12}$$

Placing $f_k=a\sigma$, $2n\lambda=A$, and $ac_1=B$, we finally obtain

$$d\epsilon/dt=A\sinh B\sigma, \tag{13}$$

where σ = the stress in the specimen in the neighborhood of the kth plane.

Equation (13) is of the form which has been proposed by several workers[2-4] for the description of plastic deformational phenomena in metals as well as high polymers. While the foregoing theory itself gives no description of the elementary mechanism of dislocation, Assumptions I and II may be regarded as establishing conditions which must be met by postulated molecular forces in a more detailed theory leading to Eq. (13).

[1] Cf., for example, Max Planck, *Theory of Heat* (MacMillan Company, Ltd., London, 1932), p. 227.
[2] A. Nadai, *Stephen Timoshenko Anniversary Volume* (MacMillan Company, Ltd., London, 1938), p. 155.
[3] A. Tobolsky and H. Eyring, J. Chem. Phys. **11**, 125 (1943).
[4] A. V. Tobolsky and R. D. Andrews, J. Chem. Phys. **13**, 3 (1945).

Discrete Space-Time and Integral Lorentz Transformations

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December 22, 1947

THE idea of introducing discreteness into space and time has occasionally been considered[2] as a means of removing the "infinities" which trouble modern physical theory, both classical and quantal. The objection which is usually raised against such discrete schemes is that they are not invariant under the Lorentz group. The purpose of this investigation is to show that *there is a simple model of discrete space-time which, although not invariant under all Lorentz transformations, does admit a surprisingly large number of Lorentz transformations.* This group of transformations is, in fact, sufficiently large to make conceivable the use of this model as a background for physical theory.

Consider all events in Minkowski space-time whose four coordinates $x^r\equiv(t, x, y, z)$ are integers.[3] (The velocity of light is taken as unity.) We shall call the set of such events the "hypercubic lattice," or *cubic lattice,* for short. The proper homogeneous Lorentz transformations which leave the cubic lattice as a whole invariant will be called *integral Lorentz transformations.* A world vector whose four coordinates are integers will be called an *integral vector*; the adjective *primitive* will be added if the components of the vector have no common integer factor other than ± 1. Two integral vectors obtainable from one another by an integral Lorentz transformation will be called *integral transforms.* We now give a brief sketch of the principal results; a more detailed account will be published elsewhere.

The integral Lorentz transformations form a group. A Lorentz transformation $x'^r=L_s{}^r x_s$ is integral if and only if all its components $L_s{}^r$ are integers. Thus the problem of finding all integral Lorentz transformations is equivalent to the solution of 10 quadratic Diophantine equations in 16 unknown integers $L_s{}^r$. This problem can be approached by means of the two-dimensional spinor calculus[4] and the elementary theory of Gaussian integers.[5]

With an integral vector (t, x, y, z) we associate a Hermitian spintensor $a^{\dot\alpha\beta}$ as follows:

$$\begin{aligned}a^{\dot 1 1}&=t+z, & a^{\dot 1 2}&=x-iy,\\ a^{\dot 2 1}&=x+iy, & a^{\dot 2 2}&=t-z.\end{aligned} \tag{1}$$

It can be shown that all primitive integral null vectors pointing into the future are represented by spinvectors c^{α}:

$$a^{\dot{\alpha}\beta}=c^{\dot{\alpha}}c^{\beta}, \qquad (2)$$

where c^1, c^2 are Gaussian integers, so that c^1 and c^2 are relatively prime and neither is divisible by $1+i$, or else so that $1+i$ is the greatest common factor of c^1, c^2 and one of them is divisible by 2. Such spinvectors will be called *integral.*

As is well known, the spin transformations

$$c'^{\alpha}=\lambda_{\beta}{}^{\alpha}c^{\beta}, \quad c'^{\dot{\alpha}}=\bar{\lambda}_{\dot{\beta}}{}^{\dot{\alpha}}c^{\dot{\beta}}, \quad |det(\lambda_{\beta}{}^{\alpha})|=1, \qquad (3)$$

represent all proper homogeneous Lorentz transformations, $\lambda_{\beta}{}^{\alpha}$ being determined by $L_s{}^r$ to within an arbitrary phase factor $e^{i\theta}$. If this phase factor is chosen suitably, the integral Lorentz transformations are represented by exactly those spin transformations which, together with their inverse, map integral spinvectors into integral spinvectors. This theorem enables us to find the following spin representation of the integral Lorentz group:

$$\lambda_1{}^1\lambda_2{}^2-\lambda_2{}^1\lambda_1{}^2=1, \qquad (4)$$

where one of the following cases applies: I. $\lambda_{\beta}{}^{\alpha}$ are Gaussian integers such that $\Sigma_{\alpha\beta}\lambda_{\beta}{}^{\alpha}$ is divisible by $1+i$. II. $\lambda_{\beta}{}^{\alpha}=\mu_{\beta}{}^{\alpha}/(1+i)$, where $\mu_{\beta}{}^{\alpha}$ are Gaussian integers not divisible by $1+i$. III. $\lambda_{\beta}{}^{\alpha}=2^{\frac{1}{2}}\mu_{\beta}{}^{\alpha}/(1+i)$, where $\mu_{\beta}{}^{\alpha}$ are Gaussian integers such that $\Sigma_{\alpha\beta}\,\mu_{\beta}{}^{\alpha}$ is divisible by $1+i$. IV. $\lambda_{\beta}{}^{\alpha}=\frac{1}{2}2^{\frac{1}{2}}\mu_{\beta}{}^{\alpha}$, where $\mu_{\beta}{}^{\alpha}$ are Gaussian integers not divisible by $1+i$. The theory of Gaussian integers shows immediately that each of the above cases includes an infinity of spin transformations. Thus, *the integral Lorentz group is infinite,* though discrete.

From the above it can be deduced that all primitive integral null vectors are equivalent, in the sense that any two of them are integral transforms of one another.

Consider any integral vector and form the set of all its integral transforms. Project each of these vectors onto the xyz space. Then the directions defined by these projections in 3-space are everywhere dense. This shows that our discrete space-time model possesses a large measure of *spatial isotropy.* It is obvious that our cubic lattice is invariant under all translations which map one lattice point into another. In this sense our discrete model is *homogeneous.*

Finally we must mention a property of our model which constitutes a drawback as far as hopes for physical application are concerned. The velocities associated with integral Lorentz transformations are given by the formula $v=(n^2-1)^{\frac{1}{2}}/n$, where n is any positive integer. The smallest non-zero velocity is $\frac{1}{2}3^{\frac{1}{2}}=0.866$ times the velocity of light.

[1] Frank B. Jewett Fellow, on leave from Carnegie Institute of Technology, Pittsburgh, Pennsylvania.
[2] V. Ambarzumian and D. Iwanenko, Zeits. f. Physik **64**, 563 (1930); L. Silberstein, *Discrete Space-Time* (University of Toronto Studies, Physics Series, 1936).
[3] The coordinates are integral multiples of a "fundamental length" ϵ (probably of the order of nuclear dimensions). We choose ϵ as the unit of length.
[4] O. Laporte and G. E. Uhlenbeck, Phys. Rev. **37**, 1381 (1931).
[5] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford University Press, New York, 1908), Chapter XII.

Range and Energy of Beta-Radiation from Calcium 45

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THE radiations given off by Ca^{45} have been investigated by Walke, Thompson, and Holt[1] who report beta-radiations of maximum energy 0.2 and 0.9 Mev, gamma-radiation of 0.7 Mev, and a half-life of 180 days. The energies of the radiation were determined by absorption measurements. In view of the importance of this isotope in biological research, it has seemed advisable to reinvestigate the radiation characteristics.

Experimental details.—Carrier-free Ca^{45}, produced by $n-p$ reaction on monoisotopic Sc^{45}, was obtained from the Atomic Energy Commission. The counting apparatus was the same as that previously described;[2] a thin-window (1.9 mg/cm²) Geiger counter was used. The source was essentially carrier-free, and was deposited in a thin (0.017-inch) aluminum stamping. The beta-radiation was measured by absorption in aluminum foils.

Results and discussion.—The method of Feather[3] was used in analyzing the results, as previously described.[2] The initial strength of the sources varied from 3000 to

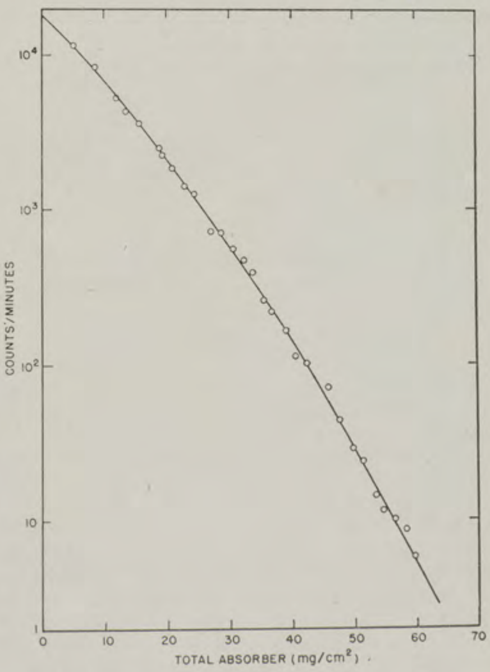

FIG. 1. Aluminum absorption curve of Ca^{4} β-radiation.