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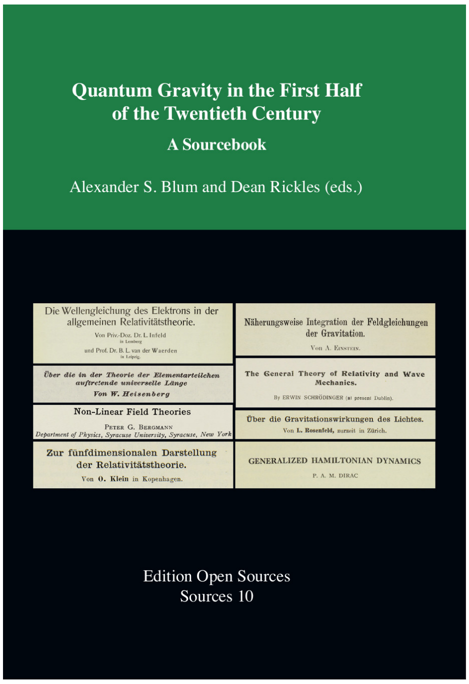
DOI: 10.34663/9783945561317-24



In: Alexander S. Blum and Dean Rickles (eds.): *Quantum Gravity in the First Half of the Twentieth Century : A Sourcebook*
Online version at https://edition-open-sources.org/sources/10/

ISBN 978-3-945561-31-7, DOI 10.34663/9783945561317-00
First published 2018 by Max-Planck-Gesellschaft zur Förderung der Wissenschaften, Edition Open

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de

Chapter 22
Paul Weiss (1938): On the Hamilton-Jacobi Theory and Quantization of a Dynamical Continuum

Paul Weiss (1938). On the Hamilton-Jacobi Theory and Quantization of a Dynamical Continuum. *Proceedings of the Royal Society of London*, A169, 102–119.

On the Hamilton-Jacobi theory and quantization of a dynamical continuum

By P. Weiss, Ph.D., *Downing College, Cambridge*

(*Communicated by R. H. Fowler, F.R.S.—Received* 1 *September* 1938)

1. Introduction

The quantization of the dynamics of point systems is closely connected with the Hamilton-Jacobi theory of the calculus of variations for *simple* integrals, the latter being a suitable mathematical formalism for describing the classical laws of point dynamics. This connexion is evident from the fact that, in order to quantize a dynamical system, one has first to know what the *pairs of canonically conjugate variables* are, and also from the fact that the quantum laws, if expressed in terms of commutation brackets, have exactly the same form as the classical laws, if expressed in terms of *Poisson brackets*.

In dealing with the quantization of the dynamics of continuous media, e.g. the electromagnetic and other fields, which has been developed along different lines, following Heisenberg and Pauli (1929), it seems tempting to try to base the method of quantization on an extended Hamilton-Jacobi theory of the calculus of variations for *multiple* integrals, the latter being the appropriate formalism for describing most relevant systems of continuum physics.

In a previous paper (Weiss 1936), referred to as I, such an attempt has been made, by extending the notion of pairs of conjugate variables. That attempt led to quantum relations on an arbitrary space-like section in space-time, provided that one *postulated* that, if the space-like section becomes especially a space section, the quantum relations should go over into those of Heisenberg and Pauli.

In the present paper the notion of Poisson brackets will be extended as well, and it will be *deduced* that the classical laws of a dynamical continuum (which satisfies a variation principle), expressed in terms of Poisson brackets, have exactly the same form as the quantum laws arrived at in I. The procedure of quantizing a dynamical continuum therewith becomes exactly the same as the procedure of quantizing point dynamics, and it will yield all physically relevant results of I.

The main part of this paper will be concerned with classical considerations, i.e. the Hamilton-Jacobi theory of continua leading from Euler's partial differential equations to Poisson brackets. The transition to the quantum theory of continua will then be a matter of a few lines. The argument, which is modelled on the Hamilton-Jacobi theory of mechanics, starts from the "complete variation" and "boundary formula", like the argument of I, and will then lead on to an extension of Poincaré's "relative integral invariant", from there to Lagrange brackets and then to Poisson brackets.

To obtain Poincaré's integral invariant and the subsequent notions, it will be necessary to use a representation of all quantities in terms of *initial data*. The use of such a representation presupposes that the state of the dynamical continuum in space-time is uniquely determined by initial data (plus boundary data in space) and that it depends continuously on these data. In other words, the initial value problem must be *correctly set*.* This is true only for theories describing a *dynamical* continuum, not, for instance, for equilibrium theories. Thus we have the satisfactory feature that the method of quantization is for purely mathematical reasons restricted to dynamical theories, as it should be on obvious physical grounds.

The initial data need not be given on a space section ($t=$ const.), but may be given on any space-like section. Mathematically, this is well known from the theory of partial differential equations. The physical implications of this fact are that the present formalism will be relativistically invariant throughout. This is an advantage as compared with the work of Heisenberg and Pauli, where a space section was fixed once and for all, in consequence of which the relativistic invariance of the result was not apparent and had to be proved separately.

Another implication relates to the notion of "state" in continuum dynamics. The word "state" in the classical sense (cf. Cartan 1922, p. 4) is synonymous with "correctly set data", and it is clear from the above remarks that it will be relativistic. It is also clear, from the meaning of the word "data", that though referring only to a space-like cross-section, it describes the motion of the dynamical continuum throughout space-time. The same is true for the word "state" in the quantum sense (Dirac 1935, ch. I), meaning the maximum information of non-contradictory data relating to the dynamical system. For the fundamental difference between the classical and the quantum sense of the word state does not affect the question of its relativistic invariance. Thus by allowing space-like sections

* For a detailed discussion of the notion of "correctly set data" in the theory of partial differential equations, cf. Hadamard (1923, 1932, ch. I) and Courant-Hilbert (1937, chs. III, V).

as bearers of the data, and not restricting oneself to space sections of constant time, the two meanings of the word state, distinguished by Dirac both in classical and quantum physics, can be reconciled with each other.

Some basic considerations of I will be repeated. As in I, the continuum will be n-dimensional and will be described by ν dependent variables. The notions of "space-like" and "time-like" can still be defined mathematically for an n-dimensional space-time, through the theory of differential equations of hyperbolic type. This generality serves to accentuate the mathematical character of our procedure. The comparison with the case of mechanics, $n=1$, will be discussed at different stages of the argument. The mathematical tool applicable to the general case, $n>1$, is furnished by the theory of functionals.*

The present considerations are not immediately applicable to physics, since all physical continuum theories, like electrodynamics, have special features requiring special discussions.† We shall deal with these questions in a later communication where the present considerations will serve as a suitable basis.

2. Action integral

We consider a system with n independent variables x^i and ν dependent variables z^α, whose classical laws satisfy a variation principle. The derivatives $\dfrac{\partial z^\alpha}{\partial x^k}$ will be denoted by z^α_k. Let

$$\mathsf{I} = \int_D^{(n)} \mathscr{L}\{x^i, z^\alpha(x), z^\alpha_k(x)\}\, dx \quad \begin{pmatrix} i, k=1, 2, \ldots, n \\ \alpha=1, 2, \ldots, \nu \end{pmatrix} \tag{2·1}$$

be the action integral.‡ It is an n-fold integral extended over a domain D of the n-dimensional "x-space". dx stands for the product $dx^1 dx^2 \ldots dx^n$. The boundary of D, which forms an $(n-1)$-dimensional closed manifold,

* Cf. Volterra (1913, 1930, 1937), De Donder (1935), Prange (1915), and Juvet (1926). The present considerations resemble more closely Prange's work than that of any other author. Many of the notions used here have been defined by Prange for the case of a two-dimensional continuum. He did not, however, consider the question of correctly set data, so that his arguments referring to Poincaré's integral invariant, etc. are merely formal and his analogies with mechanics not quite correct.

Poisson brackets have first been introduced in the theory of functionals by Volterra (1913, p. 74), but without reference to the calculus of variations.

† In I, § 5, the special features of electrodynamics and their consequences have been discussed in detail.

‡ Throughout this paper Greek indices will vary from 1 to ν, Latin indices $i, j, k, \ldots$ from 1 to n, Latin indices $r, s, \ldots$ from 1 to $n-1$. We shall use the customary summation convention.

will be denoted by S and be referred to $n-1$ parameters u^r. The integrand $\mathscr{L}$ is the Lagrangian function which characterizes the particular system under consideration.

In the case $n=1$ and if x denotes the time, we have a mechanical system of ν degrees of freedom. The boundary S degenerates in this case into two distinct points and the action integral becomes

$$\mathsf{I} = \int_1^2 \mathscr{L}\{x, z^\alpha(x), z'^\alpha(x)\}\, dx. \tag{2·1a}$$

This difference between the cases $n=1$ and $n>1$ is responsible for some difficulties which arise if one tries to generalize the theory of Hamiltonian mechanics to the case of continuous media.

It will sometimes be convenient to picture the functions $z^\alpha(x)$ as an n-dimensional surface in the $(n+\nu)$-dimensional "(x,z)-space". Its $(n-1)$-dimensional closed boundary may still be referred to the parameters u^r of S; for S is the projection of that boundary into the x-space.

3. Complete variation

We define variations η^α by varying the functions $z^\alpha(x)$,

$$\bar{z}^\alpha(x) = z^\alpha(x) + \epsilon\,\eta^\alpha(x), \tag{3·1}$$

and variations δx^i, δz^α by

$$\bar{x}^i = x^i + \epsilon\,\delta x^i(x), \quad \bar{z}^\alpha(\bar{x}) = z^\alpha(x) + \epsilon\,\delta z^\alpha(x), \tag{3·2}$$

ϵ being an infinitesimal parameter.

These variations are connected by the relation

$$\delta z^\alpha = \eta^\alpha + z^\alpha_l\,\delta x^l. \tag{3·3}$$

The derivatives $\dfrac{\mathfrak{d}\eta^\alpha}{\mathfrak{d}x^k}$ will be denoted by η^α_k.

If we picture our quantities in the (x,z)-space, the variations δx^i, δz^α are more fundamental than the η^α, because $(\epsilon\,\delta x^i, \epsilon\,\delta z^\alpha)$ is the displacement of the point (x^i, z^α) in the (x,z)-space.

By the "*complete variation*" $\delta\mathsf{I}$ we understand the variation of the action integral I due to the variations (3·1) and (3·2). We obtain for it

$$\delta\mathsf{I} = \int_D^{(n)} \left(\frac{\partial\mathscr{L}}{\partial z^\alpha}\eta^\alpha + \frac{\partial\mathscr{L}}{\partial z^\alpha_k}\eta^\alpha_k\right) dx + \oint_S^{(n-1)} \mathscr{L} N_i\,\delta x^i\, du,$$

where the N_i are the $(n-1)$-rowed minors of the Jacobian matrix*

$$\left\| \frac{\partial x^i}{\partial u^r} \right\|;$$

they form a vector field on S, normal to S. du stands for the product $du^1\, du^2 \ldots du^{n-1}$.

With the help of the formula

$$\frac{\partial \mathscr{L}}{\partial z^\alpha_k}\eta^\alpha_k = -\left(\frac{\partial}{\partial x^k}\frac{\partial \mathscr{L}}{\partial z^\alpha_k}\right)\eta^\alpha + \frac{\partial}{\partial x^k}\left(\frac{\partial \mathscr{L}}{\partial z^\alpha_k}\eta^\alpha\right)$$

we perform an integration by parts, using Green's theorem, and introduce on the boundary the variations δx^i, δz^α instead of η^α, by means of (3·3). $\delta\mathsf{I}$ then becomes

$$\delta\mathsf{I} = \int_D^{(n)} \left(\frac{\partial \mathscr{L}}{\partial z^\alpha} - \frac{\partial p^l_\alpha}{\partial x^l}\right)\eta^\alpha dx + \oint_S^{(n-1)} (X_i \delta x^i + P_\alpha \delta z^\alpha)\, du, \tag{3·4}$$

where we have introduced the notations†

$$p^i_\alpha = \frac{\partial \mathscr{L}}{\partial z^\alpha_i}, \quad U^i_k = \mathscr{L}\delta^i_k - p^i_\beta z^\beta_k, \tag{3·5}$$

$$P_\alpha = p^l_\alpha N_l, \quad X_k = U^l_k N_l. \tag{3·6}$$

The ν coefficients of η^α under the n-fold integral on the right-hand side of (3·4) are the Eulerian expressions. If they vanish, we have the Eulerian equations

$$\frac{\partial \mathscr{L}}{\partial z^\alpha} - \frac{\partial p^l_\alpha}{\partial x^l} = 0. \tag{3·7}$$

In physics these equations constitute the classical laws. Their solutions are the "extremals" of the calculus of variations.

In the case of mechanics, $n = 1$, the Eulerian equations are the equations of motion. In the case of a continuum, $n > 1$, they are one set of field equations, another set being given by the integrability conditions which the $z^\alpha_k(x)$ must satisfy in order to be representable as derivatives $\frac{\partial z^\alpha}{\partial x^k}$; these read

$$\frac{\partial z^\alpha_k}{\partial x^i} - \frac{\partial z^\alpha_i}{\partial x^k} = 0. \tag{3·8}$$

From now onwards we shall assume that (3·7) and (3·8) are satisfied.

* The quantities N_i have in I been denoted by X_i. We have changed our notation in a few details in the hope that the present notation will be more suggestive.

† The quantities X_i have in I been denoted by σ_i and the P_α by π_α.

4. Boundary formula and conjugate variables

It follows from (3·4) that extremals, instead of being characterized by (3·7), may alternatively be characterized by the equation

$$\delta \mathsf{I} = \oint_S^{(n-1)} (X_i \delta x^i + P_\alpha \delta z^\alpha)\, du. \tag{4·1}$$

We shall call this equation the "*boundary formula*".* The action integral I may be considered as a functional of the boundary—$x^i(u)$, $z^\alpha(u)$—in the (x, z)-space and the boundary formula (4·1) as its differential. X_i and P_α are therefore the functional derivatives of I with respect to x^i and z^α respectively, taken at a point (u) of the boundary. By this property X_i and P_α are defined as the conjugates to x^i and z^α so that we have the "*pairs of conjugate variables*"

$$(X_i, x^i), \quad (P_\alpha, z^\alpha). \tag{4·2}$$

In the case of mechanics the boundary formula becomes

$$\delta \mathsf{I} = \left| \mathscr{H}\, \delta x + p_\alpha \delta z^\alpha \right|_1^2, \tag{4·1a}$$

where we have put†

$$p_\alpha = \frac{\partial \mathscr{L}}{\partial z'^\alpha}, \quad \mathscr{H} = \mathscr{L} - p_\beta z'^\beta \tag{3·5a}$$

and inverted the sign of the variations at the point 1. $\mathscr{H}$ represents the negative energy of the system. It will be noticed that (3·5) and (3·6) *both* reduce to (3·5a).

I degenerates into a "two-point function" of the co-ordinates (x^i, z^α) of the two boundary points in the (x, z)-space. The pairs

$$(\mathscr{H}, x), \quad (p_\alpha, z^\alpha) \tag{4·2a}$$

are the well-known conjugate variables of Hamiltonian mechanics. This fact shows that our present considerations may be considered as a generalization of Hamiltonian mechanics.

It should be noted that in the general case, $n > 1$, the conjugates of x^i and z^α respectively depend not only on a point in the x-space, like the z^α, but also on the *direction* of a surface element passing through this point. This fact will be important in our subsequent considerations.

* This term is the translation of the French "formule aux limites". Cf. Hadamard (1910, pp. 142–51).

† Our definition of $\mathscr{H}$ differs from the usual one in sign.

5. Natural co-ordinate system and Legendre transformation

A "*natural co-ordinate system*" is defined in the x-space and is adapted to the boundary S in such a way that on S the first co-ordinate, say w, is normal to S and the other $n-1$ co-ordinates are the parameters u^r of S. We shall use the notation

$$z'^{\alpha} = \frac{\mathfrak{d} z^{\alpha}}{\mathfrak{d} w}, \quad \tilde{z}^{\alpha}_{r} = \frac{\mathfrak{d} z^{\alpha}}{\mathfrak{d} u^r}. \tag{5·1}$$

For the conjugates P_{α}, X_k, defined by (3·6), we obtain in a natural co-ordinate system

$$P_{\alpha} = \frac{\partial \mathscr{L}}{\partial z'^{\alpha}}; \quad X_1 = \mathscr{L} - P_{\beta} z'^{\beta}, \quad X_{1+r} = -P_{\beta} \tilde{z}^{\beta}_{r}. \tag{5·2}$$

Let us put $$\mathscr{H} = \mathscr{L} - P_{\beta} z'^{\beta}, \quad h_r = -P_{\beta} \tilde{z}^{\beta}_{r}. \tag{5·3}$$

The boundary formula (4·1) may now be written as

$$\delta \mathsf{I} = \oint_{S}^{(n-1)} (\mathscr{H}\, \delta w + h_r \delta u^r + P_{\alpha} \delta z^{\alpha})\, du. \tag{5·4}$$

By means of the *Legendre transformation* (Courant-Hilbert 1937, p. 26)

$$z'^{\alpha} \to P_{\alpha}, \quad \mathscr{L}(z^{\alpha}, \tilde{z}^{\alpha}_{r}, z'^{\alpha}) \to \mathscr{H}(z^{\alpha}, \tilde{z}^{\alpha}_{r}, P_{\alpha}) \tag{5·5}$$

we replace the derivatives z'^{α} normal to S by the P_{α}, and the classical laws can now be written in the "*canonical form*"

$$\frac{\partial \mathscr{H}}{\partial z^{\alpha}} - \frac{\mathfrak{d}}{\mathfrak{d} u^r} \frac{\partial \mathscr{H}}{\partial \tilde{z}^{\alpha}_{r}} - \frac{\mathfrak{d} P_{\alpha}}{\mathfrak{d} w} = 0, \quad \frac{\partial \mathscr{H}}{\partial P_{\alpha}} + \frac{\mathfrak{d} z^{\alpha}}{\mathfrak{d} w} = 0. \tag{5·6}$$

In the case $n=1$ the question of the natural co-ordinate system does not arise, $\mathscr{H}$ goes over into the Hamiltonian of mechanics, and the canonical equations take the well-known form

$$\frac{\partial \mathscr{H}}{\partial z^{\alpha}} - \frac{dp_{\alpha}}{dx} = 0, \quad \frac{\partial \mathscr{H}}{\partial p_{\alpha}} + \frac{dz^{\alpha}}{dx} = 0. \tag{5·6a}$$

Thus the function $\mathscr{H}$ defined by (5·3) appears as the most natural generalization of the Hamiltonian of mechanics, though other generalizations have also been given.*

Although $\mathscr{H}$ has been introduced as a component—X_1—of a vector in the natural co-ordinate system, it has an invariant significance, because for a given surface element of S the co-ordinate w normal to S has an invariant significance, and $\mathscr{H}$ is the conjugate to this co-ordinate w.

* The best known one is $\mathscr{H} = \mathscr{L} - p^{i}_{\beta} z^{\beta}_{i}$, first given by Volterra.

6. Boundary and initial data. The hyperbolic type

So far we have considered the extremal surfaces $z^\alpha(x)$ as given by their *boundary* $x^i(u)$, $z^\alpha(u)$ in the (x, z)-space, or in other words, by prescribing the "*boundary data*" $z^\alpha(u)$ on the whole boundary S in the x-space. Generally speaking, one understands by boundary data the prescribing of *one* quantity, e.g. z^α *or* P_α, on the whole of S.

For many considerations, however, in particular for deriving Poincaré's integral invariant, Lagrange brackets and Poisson brackets, it is necessary to characterize extremals by "*initial data*". Here we understand by initial data that on a part of the boundary S *two* data, e.g. z^α *and* P_α, are prescribed, while on another part of S no datum is prescribed.

In mechanics an extremal may in general be given by initial data equally well as by boundary data. In either case it is uniquely determined by its data* and depends continuously on them; the same is therefore true of the action integral I. There is thus no difficulty in passing over from the one standpoint to the other, as is done in the Hamilton-Jacobi theory of mechanics.

In a continuum theory the position is more complicated. One can distinguish between different types of systems according to what kind of data are "*correctly set*". This term has been explained in the introduction. There is one type of systems, the so-called "*elliptic type*", for which boundary data are in general correctly set, but initial data are not. In continuum physics all *equilibrium problems* belong to this type.

All physical problems of continuum *dynamics* belong to another type, called the "*hyperbolic type*". Before we can state precisely what kind of data are correctly set for this type, we have to give some preliminary explanations. For a detailed treatment we refer to the works of Hadamard and Courant, quoted in the introduction. There exists for the hyperbolic type a *real* $(n-1)$-dimensional "*characteristic cone*" (in physics called "light cone") at each point of the x-space, which gives rise to a classification of a given closed boundary S into different regions. For $n > 2$ there are regions on S, called *space-like*, where the characteristic cone at a given point P of S intersects S only in P itself and in no other point of a suitable neighbourhood of P; there are other regions on S, called *time-like*, where the cone at P intersects S in an infinity of points in any given neighbourhood of P. The frontier between space-like and time-like regions on S is an $(n-2)$-dimensional manifold of points whose characteristic cones *touch* S. For $n = 2$,

* The case where this is not true, e.g. "conjugate points", arises from an exceptional choice of data.

there are still the same different regions on S, but their roles of being space- or time-like may be interchanged, the characteristic cone degenerating into a pair of lines passing through P.

In physics, the history of a domain of space during an interval of time is described by a region D in space-time ($n=4$) of cylindrical shape. The boundary S of D consists of two bases, which are space-like, and of one curved part representing the history of the boundary of the domain of space, which is time-like. The principle of general relativity permits all those transformations of space-time which preserve the character of a region of being space- or time-like. Any closed surface S consisting of two distinct space-like regions, S_1 and S_2 say, and of a time-like region, joining S_1 with S_2, is relativistically equivalent to a cylinder of the kind described above; it is therefore admissible as a boundary for a problem of continuum dynamics.

For all problems of the hyperbolic type boundary data throughout S are in general not correctly set. The only correctly set data are: *initial data, z^α and P_α, on one of the two space-like regions of S, S_1 say; boundary data, z^α or P_α, on the time-like region; no data on the remaining space-like region, S_2 say.* Such data are called "*mixed data*".

For $n > 3$, there exist still other types of systems, called "ultrahyperbolic", for which neither boundary data nor mixed data are in general correctly set. No physical problem belongs to such a type.

From now on we shall deal with the hyperbolic type only.

7. Interpretation of the boundary formula for the hyperbolic type

It is clear from the preceding considerations that the notions of integral invariant, Lagrange brackets and Poisson brackets can be extended only to continuous systems of the hyperbolic type and only as far as the initial data, i.e. the data on a space-like region are concerned. This, however, is precisely what physics requires.

On the other hand, the boundary formula (4·1) has only a formal meaning in the case of continuum dynamics, since the functional I does not depend uniquely and continuously on the boundary data. The conclusion which we have drawn from it, i.e. which quantities are pairs of conjugate variables, is also of a merely formal kind and is therefore valid for any type of system. But for our further considerations we have to adapt the boundary formula to the fact that I is now a proper functional of the mixed data and not of the boundary data on S.

It would be difficult to obtain an expression for the general functional differential of I, describing the variation of I corresponding to the variations of the mixed data. For our purpose, however, we need only study the special case where the boundary data on the time-like region are kept fixed and the initial data on the space-like region S_1 alone are varied. The boundary formula may then be written as

$$\delta \mathsf{I} = \left| \int^{(n-1)} (X_i \delta x^i + P_\alpha \delta z^\alpha)\, du \right|_{S_1}^{S_2}, \tag{7·1}$$

where we have inverted the sign of the variations on S_1. This formula should be compared with the corresponding formula (4·1 a) for the case of mechanics.

In (7·1), the variations on S_2 are determined by the initial data on S_1 and their variations. The right-hand side of (7·1) is, therefore, not a proper differential of the functional I considered as a functional of the initial data. In the following section we shall turn this fact to our advantage.

The fixed boundary data on the time-like region represent spatial boundary conditions holding throughout time. Their physical significance depends on the particular problem under consideration. For vibrations of a string or of a membrane they are conditions imposed on the boundary of the string or membrane (as is obvious). For a pure radiation field of finite extension, enclosed between walls, they represent the optical properties of the walls. For a radiation field extending through the whole of space, they represent conditions for the behaviour of the field at infinity in space. If the field contains charged particles, forming singular time-like world-lines extending in time (more precisely: proper-time) from the infinite past to the infinite future, one may exclude these singular lines by small tubes. One then obtains a multiply connected space-time, for which the boundary data on the time-like region consist not only of conditions at infinity but also of conditions on the surfaces of the small tubes. These latter conditions describe the behaviour of the singularities.

We need not deal with these conditions in the present work, since they are not of the nature of initial conditions and thus do not affect the Hamilton-Jacobi theory and quantization of a dynamical continuum, considered by itself. They may be of importance if one considers interaction processes, for instance the interaction of a radiation field with charged particles.

8. Integral invariant and Lagrange brackets

Let us consider a one-parametric *closed* family of extremals, labelled by a parameter λ, $0 \leqq \lambda \leqq 1$, such that the time-like parts of the boundaries are the same for the whole family. The initial data on S_1 then become functions of λ,

$$S_1: x^i = x^i(u, \lambda), \quad z^\alpha = z^\alpha(u, \lambda), \quad P_\alpha = P_\alpha(u, \lambda). \tag{8·1}$$

The fact that the family is closed is expressed by the relations

$$x^i(u, 0) = x^i(u, 1), \quad z^\alpha(u, 0) = z^\alpha(u, 1), \quad P_\alpha(u, 0) = P_\alpha(u, 1).$$

We thus obtain a tube of extremals in the (x, z)-space, which is completely determined by the tube of $(n-1)$-dimensional surfaces (8·1) and by the constant time-like boundary part. The functional I thus becomes a function of the parameter λ.

Let us now apply the modified boundary formula (7·1) to this tube by putting

$$\delta x^i = \frac{dx^i}{d\lambda} d\lambda, \quad \delta z^\alpha = \frac{dz^\alpha}{d\lambda} d\lambda$$

and then integrate it round the tube, i.e. from 0 to 1. The result must be zero since

$$\mathsf{I}(0) = \mathsf{I}(1).$$

Hence we obtain

$$\int_0^1 d\lambda \int_{S_1(\lambda)}^{(n-1)} \left(X_i \frac{dx^i}{d\lambda} + P_\alpha \frac{dz^\alpha}{d\lambda} \right) du = \int_0^1 d\lambda \int_{S_2(\lambda)}^{(n-1)} \left(X_i \frac{dx^i}{d\lambda} + P_\alpha \frac{dz^\alpha}{d\lambda} \right) du. \tag{8·2}$$

The formula (8·2) holds for any two tubes of $(n-1)$-dimensional space-like surfaces $S_1(\lambda)$ and $S_2(\lambda)$, provided that they both lie on the same tube of extremals. Hence the expression in (8·2) is invariant under a displacement along the tube. We have thus obtained a generalization of the notion of "*relative integral invariant*".*

$$\int_0^1 d\lambda \int_{S_1(\lambda)}^{(n-1)} \left(X_i \frac{dx^i}{d\lambda} + P_\alpha \frac{dz^\alpha}{d\lambda} \right) du. \tag{8·3}$$

* Cf. Cartan (1922, pp. 4, 9), Whittaker (1927, p. 272). Recently Born (1934) has attempted to derive a quantization method from Hilbert's "*independent integral*" (Hilbert 1900, 1905). Hilbert's integral can be given the same form as that of (8·3), but it applies to *selections* (Hilbert calls them "fields") of extremals such that through each point in the (x, z)-space there passes one and only one extremal. Hilbert's "*independence theorem*" is an immediate consequence of the invariance of (8·3). For if a field of extremals is especially chosen in such a way that the closed integral (8·3) vanishes for all tubes $S_1(\lambda)$ on a cross-section of the (x, z)-space, it must, owing to its invariance, do so throughout the field. This is Hilbert's theorem.

Since a displacement within the same extremal gives only a trivial contribution, we may without loss of generality assume that $\frac{dx^i}{d\lambda}=0$; then we obtain a generalization of *Poincaré's relative integral invariant*,

$$\int_0^1 d\lambda \int_{S_1(\lambda)}^{(n-1)} P_\alpha \frac{dz^\alpha}{d\lambda} du. \tag{8·4}$$

The P_α and z^α appearing under the integral sign in (8·4) are the initial data which determine the extremal if the boundary data on the time-like part of S are fixed.

The relative integral invariant (8·3) or (8·4) can be transformed into an *absolute integral invariant* with the help of Stokes' theorem. Consider a two-parametric family of extremals, labelled by the parameters ω_1, ω_2 such that the closed one-parametric family forms its boundary. It may be pictured as a "solid" tube of extremals in the (x, z)-space with the given tube as its surface. We then obtain for the integral invariants (8·3) and (8·4) respectively, with an obvious notation,

$$\iint d\omega_1 d\omega_2 \int_{S_1(\omega)}^{(n-1)} \left(\frac{d(X_i, x^i)}{d(\omega_1, \omega_2)} + \frac{d(P_\alpha, z^\alpha)}{d(\omega_1, \omega_2)} \right) du, \tag{8·5}$$

$$\iint d\omega_1 d\omega_2 \int_{S_1(\omega)}^{(n-1)} \frac{d(P_\alpha, z^\alpha)}{d(\omega_1, \omega_2)} du. \tag{8·6}$$

This is the generalization of Poincaré's absolute integral invariant (cp. Cartan 1922, pp. 19–20; Whittaker 1927, p. 272).

Again, without loss of generality we may work with the expression (8·6). As the domain of integration in the parameter space is arbitrary, the integrand under the integral-sign over ω must itself remain invariant under a displacement along the extremals. This integrand is a *generalized Lagrange bracket* (it will be referred to as L.B.) (cp. Whittaker 1927, p. 298),

$$\{\omega_1, \omega_2\} = \int_{S_1}^{(n-1)} \frac{d\{P_\alpha(u), z^\alpha(u)\}}{d(\omega_1, \omega_2)} du. \tag{8·7}$$

In the case $n=1$ the space-like region S_1 consists of a single point so that (8·7) reduces to the ordinary L.B., involving no integration but only the summation over α. In the general case we have the integration over S_1 besides the summation over the number of dependent variables.

In all the subsequent considerations the variables (u^r) will appear on the same footing as the index α. It is therefore useful to consider a point (u) of S_1 as a *continuous index*, which together with the discrete index α serves to label the dependent variables z and their conjugates P. From this standpoint the only difference between the dynamics of point systems and that of continua is the larger *magnitude* of indices which in the latter case is necessary in order to specify the initial data.

This standpoint also approaches the standpoint of Heisenberg and Pauli, who treated the continuum as the limiting case of a very large number of points.

In mechanics, all transformations preserving the L.B.'s are defined as "*canonical transformations*". The same definition will be adopted for the general case. Then the invariance property of the L.B.'s may be expressed by saying that for any dynamical continuum with fixed spatial boundary conditions the motion consists of a continuous unfolding of canonical transformations, starting from some initial data on some space-like region S_1.

9. Poisson brackets

In the considerations of the preceding section we have treated the P_α and z^α as functions of the u^r on S_1 and of the parameters ω_1, ω_2. We shall now invert the connexion and treat the quantities ω_1, ω_2 as functionals of the $P_\alpha(u)$, $z^\alpha(u)$, defined on S_1. The functional differential of such a functional ω has the form

$$\delta\omega = \int_{S_1}^{(n-1)} \left(\frac{\delta\omega}{\delta P_\alpha(u)} \delta P_\alpha(u) + \frac{\delta\omega}{\delta z^\alpha(u)} \delta z^\alpha(u) \right) du, \tag{9·1}$$

where $\frac{\delta\omega}{\delta P_\alpha(u)}$ and $\frac{\delta\omega}{\delta z^\alpha(u)}$ denote the functional derivatives of ω at the point (u).

In particular, we shall have to consider the functionals $P_\beta(v)$ and $z^\beta(v)$, i.e. the values of P_β and z^β at some given point (v) on S_1. We obtain from (9·1)

$$\left.\begin{aligned} \frac{\delta P_\beta(v)}{\delta P_\alpha(u)} &= \delta^\alpha_\beta \, \delta(v-u), & \frac{\delta P_\beta(v)}{\delta z^\alpha(u)} &= 0, \\ \frac{\delta z^\beta(v)}{\delta P_\alpha(u)} &= 0, & \frac{\delta z^\beta(v)}{\delta z^\alpha(u)} &= \delta^\beta_\alpha \, \delta(v-u), \end{aligned}\right\} \tag{9·2}$$

where $\delta(v-u)$ denotes the product of the $n-1$ δ-functions $\delta(v^r-u^r)$ of Dirac.

For any two functionals ω_1, ω_2 we define the *generalized Poisson bracket* (referred to as P.B.) (cp. Whittaker 1927, p. 299) by

$$[\omega_1, \omega_2] = \int_{S_1}^{(n-1)} \frac{d(\omega_1, \omega_2)}{d\{P_\alpha(u), z^\alpha(u)\}} du, \tag{9·3}$$

where the Jacobian under the integral sign is defined as

$$\frac{d(\omega_1, \omega_2)}{d\{P_\alpha(u), z^\alpha(u)\}} = \frac{\delta\omega_1}{\delta P_\alpha(u)}\frac{\delta\omega_2}{\delta z^\alpha(u)} - \frac{\delta\omega_1}{\delta z^\alpha(u)}\frac{\delta\omega_2}{\delta P_\alpha(u)}. \tag{9·4}$$

In the case of mechanics the generalized P.B.'s defined by (9·3) reduce to the ordinary P.B.'s. They also obey the same formal rules as the ordinary P.B.'s.

The P.B.'s have the same invariance properties as the L.B.'s. We shall show this by proving a relationship which allows us to express the P.B.'s in terms of the L.B.'s and vice versa. The prescribing of the initial data $P_\alpha(u)$, $z^\alpha(u)$ on S_1 (always for *fixed* boundary data on the time-like region of S) may be considered as a way of labelling all extremals (with the given fixed boundary data). There is, however, no need to choose especially the initial data for labelling the extremals. Any other set of the same magnitude may be used equally well. Let $\omega_a(v)$ $(a = 1, 2, \ldots, 2\nu;\ v \in S_1)$ be such a set which labels the extremals. We shall then have a one-one-correspondence

$$P_1(u) \ldots P_\nu(u), \quad z^1(u) \ldots z^\nu(u) \leftrightarrow \omega_1(v) \ldots \omega_{2\nu}(v), \tag{9·5}$$

which may be interpreted as a *non-singular functional transformation* between two different modes of labelling the extremals.

Taking any two quantities, $\omega_a(v)$ and $\omega_b(v')$ say, we can form their L.B. and their P.B. The general relationship which we want to prove reads

$$\int_{S_1}^{(n-1)} dv \sum_{a=1}^{a=2\nu} \{\omega_a(v), \omega_b(v')\}\,[\omega_a(v), \omega_c(v'')] = \delta_c^b\,\delta(v'-v'') \quad (b, c = 1, 2, \ldots, 2\nu).* \tag{9·6}$$

This relation expresses that the matrix formed by all the L.B.'s and the matrix formed by all the P.B.'s are contragredient (i.e. inverse and transposed) to each other. A corresponding relation holds in mechanics, with the only difference that we have here the continuous index v besides the discrete index a which results in the matrices having a continuous range of rows and columns. The proof of (9·6) therefore proceeds on the same lines as the corresponding proof in mechanics (Whittaker 1927, pp. 299–300).

To prove (9·6), we first use the definitions (8·7) and (9·3), (9·4) for the L.B.'s and P.B.'s so that (9·6) becomes

$$\int dv \sum_{a=1}^{a=2\nu} \int du \left(\frac{\delta P_\alpha(u)}{\delta\omega_a(v)}\frac{\delta z^\alpha(u)}{\delta\omega_b(v')} - \frac{\delta z^\alpha(u)}{\delta\omega_a(v)}\frac{\delta P_\alpha(u)}{\delta\omega_b(v')}\right) \times \int du' \left(\frac{\delta\omega_a(v)}{\delta P_\beta(u')}\frac{\delta\omega_c(v'')}{\delta z^\beta(u')} - \frac{\delta\omega_a(v)}{\delta z^\beta(u')}\frac{\delta\omega_c(v'')}{\delta P_\beta(u')}\right),$$

* It will be seen from the definition (8·7) of the L.B.'s that $\{\omega_a, \omega_b\}$ is contravariant in a and b. Since this fact is not expressed in our notation, we use the Σ-symbol for the summation over a.

it being understood that the integration always extends over the $(n-1)$-dimensional region S_1. In evaluating this expression we use the relations

$$\int dv \sum_{a=1}^{a=2\nu} \frac{\delta P_\alpha(u)}{\delta\omega_a(v)} \frac{\delta\omega_a(v)}{\delta P_\beta(u')} = \delta_\alpha^\beta\,\delta(u-u'), \quad \int dv \sum_{a=1}^{a=2\nu} \frac{\delta P_\alpha(u)}{\delta\omega_a(v)} \frac{\delta\omega_a(v)}{\delta z^\beta(u')} = 0,$$
$$\int dv \sum_{a=1}^{a=2\nu} \frac{\delta z^\alpha(u)}{\delta\omega_a(v)} \frac{\delta\omega_a(v)}{\delta P_\beta(u')} = 0, \quad \int dv \sum_{a=1}^{a=2\nu} \frac{\delta z^\alpha(u)}{\delta\omega_a(v)} \frac{\delta\omega_a(v)}{\delta z^\beta(u')} = \delta_\beta^\alpha\,\delta(u-u'),$$

which follow directly from (9·2). We then obtain for the above expression

$$\int du \int du' \left(\frac{\delta z^\alpha(u)}{\delta\omega_b(v')} \frac{\delta\omega_c(v'')}{\delta z^\beta(u')} \delta_\alpha^\beta\,\delta(u-u') + \frac{\delta P_\alpha(u)}{\delta\omega_b(v')} \frac{\delta\omega_c(v'')}{\delta P_\beta(u')} \delta_\beta^\alpha\,\delta(u-u') \right)$$
$$= \int du \left(\frac{\delta z^\alpha(u)}{\delta\omega_b(v')} \frac{\delta\omega_c(v'')}{\delta z^\alpha(u)} + \frac{\delta P_\alpha(u)}{\delta\omega_b(v')} \frac{\delta\omega_c(v'')}{\delta P_\alpha(u)} \right) = \delta_c^b\,\delta(v'-v''),$$

so that (9·6) is proved.

It follows that the P.B.'s are invariant under canonical transformations.

10. Classical laws in terms of Poisson brackets

As in mechanics, we can formulate explicit conditions which are necessary and sufficient for a transformation of the $P_\alpha(u)$, $z^\alpha(u)$ to be canonical. For, from the formulae (9·2)–(9·4), we obtain for trivial reasons

$$\left. \begin{aligned} [P_\alpha(v), P_\beta(v')] = 0, \quad [z^\alpha(v), z^\beta(v')] = 0, \\ [P_\alpha(v), z^\beta(v')] = \delta_\alpha^\beta\,\delta(v-v'), \end{aligned} \right\} \tag{10·1}$$

and it follows from the invariance of the P.B.'s that the transformed quantities, if obtained by a canonical transformation, must satisfy the same relations. This proves that the relations (10·1) are necessary.

By differentiating the second and the third relation of (10·1) with respect to v'^r we obtain, taking account of (5·1),

$$[z^\alpha(v), \tilde{z}_r^\beta(v')] = 0, \quad [P_\alpha(v), \tilde{z}_r^\beta(v')] = -\delta_\alpha^\beta\,\delta_r(v-v'), \tag{10·2}$$

where $\delta_r(v-v')$ stands for $\dfrac{\mathfrak{d}}{\mathfrak{d}u^r}\delta(v-v')$. Now let $\mathscr{F}(P_\alpha, z^\alpha, \tilde{z}_r^\alpha)$ be an arbitrary function defined on S_1 and put

$$\mathsf{F} = \int_{S_1}^{(n-1)} \mathscr{F}(v)\,dv, \quad (v \in S_1). \tag{10·3}$$

We can evaluate the P.B.'s of $P_\alpha(u)$ and $z^\alpha(u)$ with F with the help of (10·1) and (10·2) and obtain

$$[P_\alpha(u), \mathsf{F}] = \frac{\partial\mathscr{F}}{\partial z^\alpha} - \frac{\mathfrak{d}}{\mathfrak{d}u^s} \frac{\partial\mathscr{F}}{\partial \tilde{z}_s^\alpha}, \quad [z^\alpha(u), \mathsf{F}] = -\frac{\partial\mathscr{F}}{\partial P_\alpha}. \tag{10·4}$$

In this manner we can evaluate any P.B.'s, using solely the relations (10·1), the relations (10·2) being a consequence of (10·1). It follows, therefore, that (10·1) is also sufficient for a transformation to be canonical.

We shall now apply the general formulae (10·4) to the quantities $\mathscr{H}$ and h_r, defined in section 5. We put

$$\mathsf{H} = \int_{S_1}^{(n-1)} \mathscr{H}(v)\, dv, \quad \mathsf{h}_r = \int_{S_1}^{(n-1)} h_r(v)\, dv. \tag{10·5}$$

Applying (10·4) to the case $\mathscr{F} = h_r$, we obtain from the last set of (5·3)

$$[P_\alpha(u), \mathsf{h}_r] = \frac{\delta P_\alpha}{\delta u^r}, \quad [z^\alpha(u), \mathsf{h}_r] = \frac{\delta z^\alpha}{\delta u^r}. \tag{10·6}$$

Applying (10·4) to the case $\mathscr{F} = \mathscr{H}$ and using the canonical equations (5·6), we obtain

$$[P_\alpha(u), \mathsf{H}] = \frac{\delta P_\alpha}{\delta w}, \quad [z^\alpha(u), \mathsf{H}] = \frac{\delta z^\alpha}{\delta w}. \tag{10·7}$$

Since S_1 may be any space-like region, w may be any time-like co-ordinate.

Conversely, the canonical equations (5·6) follow from (10·7) with the help of the general formula (10·4). The equations (10·7) may therefore be considered as the formulation of the canonical equations by means of P.B.'s. In the case of mechanics they go over into the well-known formulae

$$[p_\alpha, \mathscr{H}] = \frac{dp_\alpha}{dx}, \quad [z^\alpha, \mathscr{H}] = \frac{dz^\alpha}{dx}. \tag{10·7a}$$

In the case $n > 1$ they are supplemented by the equations (10·6). (10·7) and (10·6) form together a set of covariant vector equations in the x-space.

It follows from (10·6) and (10·7) that we have for an arbitrary function $\mathscr{F}(P_\alpha, z^\alpha, z^\alpha_r)$,

$$[\mathscr{F}(u), \mathsf{H}] = \frac{\delta \mathscr{F}}{\delta w}, \quad [\mathscr{F}(u), \mathsf{h}_r] = \frac{\delta \mathscr{F}}{\delta u^r}. \tag{10·8}$$

In this way every dynamical relation of the continuum can be expressed in terms of the P.B.'s.

11. Quantization

Having formulated classical continuum dynamics in terms of Poisson brackets, the quantization can be performed in the same way as in mechanics. The continuous set of initial data $P_\alpha(u)$, $z^\alpha(u)$ becomes a continuous set of non-commuting observables, and a general argument, which is valid irrespective of whether the set is discrete or continuous, requires that the quantum P.B.'s, which are defined by the property of obeying the same formal rules as the classical P.B.'s, must be proportional to the commutation brackets. In this way we obtain ultimately (Dirac 1935, pp. 89–90)

$$[\omega_1, \omega_2] = \frac{i}{\hbar}(\omega_1\omega_2 - \omega_2\omega_1). \tag{11·1}$$

With this new interpretation of the P.B.'s the relations (10·1) and 10·2) become the "*Quantum Conditions*", the function $\mathscr{H}(P_\alpha, z^\alpha, \tilde{z}^\alpha_r)$ or better its integral H becomes the "*Quantum Hamiltonian*", and the relations (10·7) become the "*Quantum Equations of Motion*" of the quantized continuum. The Hamiltonian has an invariant significance, as we have seen at the end of section 5. The relations (10·4) and (10·6) follow from (10·1) and (10·2), and (10·8) follows from (10·6) and (10·7) by the same arguments as in the classical theory.

The quantum formalism thus obtained is the same as the one we had obtained in I (Weiss 1936, pp. 201–4) by an argument based on a comparison with the results of Heisenberg and Pauli. The present formalism applies to any space-like region S_1 and the corresponding time-like co-ordinate w normal to S_1, but it does not and, on physical grounds, should not apply to a time-like region.*

Summary

The Hamilton-Jacobi theory of point mechanics is extended to the mechanics of continuous media, following on the lines first proposed by Prange. The method is based on the equivalence between the Euler equations and the "boundary formula" (*formule aux limites*) in the calculus of variations. It is shown that the notions of Lagrange brackets and Poisson brackets can be extended, but only if the Euler equations of the continuum are of hyperbolic type. Consequently, these notions only apply to a dynamical continuum and not to equilibrium problems. Except for this restriction, the method is applicable quite generally, for linear as well as for non-linear theories. Once the dynamical laws are expressed in terms of Poisson brackets, the transition to the quantum theory can be effected by a brief, formal argument in the same way as in point mechanics.

* At the end of I, I tried to apply the quantization to a time-like region as well. This was due to my not having sufficiently taken account of the hyperbolic character of all dynamical problems. No quantum relations on a time-like region are justified.

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On the Hamilton-Jacobi theory and quantization of generalized electrodynamics

By P. Weiss, Ph.D., *Downing College, Cambridge*

(*Communicated by R. H. Fowler, F.R.S.—Received* 1 *September* 1938)

1. Introduction

In the preceding paper, which will be quoted as A, the Hamilton-Jacobi theory has been developed for a dynamical continuum of quite general type, and it has been shown that its classical laws can be expressed in terms of Poisson brackets involving pairs of canonically conjugate variables. In this way the theory of a dynamical continuum can be treated on the same lines as the dynamical theory of point systems.

This result is of importance for the study of the procedure of quantization, for it is a well-known fact that a quantum Poisson bracket which obeys the same algebraic rules as a classical Poisson bracket must be proportional to a commutation bracket (Dirac 1935, p. 89). This fact is independent of